The Dynamics of Earthquake Source

Liliana Hromadová

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Liliana Hromadová

Comenius University

Faculty of Mathematics, Physics and Informatics

Department of Astronomy, Physics of the Earth, and Meteorology

4.1.1 Physics

Supervisor: Prof. RNDr. Peter Moczo, DrSc.

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Abstract

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The presented thesis is intended as an introduction to the physics of the earthquake source focused on selected basic topics. We start with a presentation of a relatively simple concept of a spontaneous rupture propagation on a fault surface as a basic model of a source of a tectonic earthquake. Basic concepts - slip, slip rate, total traction, traction variation, and frictional strength on the faulting surface are defined. The most general form of the constitutive law - the friction law - is given and briefly characterized. We continue with presentation of the stress-strain relation, equation of motion, local form of the energy conservation law, and strain energy function assuming standard natural state in the continuum mechanics - the state characterized by a zero stress and strain. Then we present generalization of these relations and concepts for the case of the natural state characterized by a nonzero initial stress but zero strain. Such a natural state is convenient to consider as a state just prior an earthquake. We continue with a concise summary of the four basic models of the seismoactive fault and earthquake rupture (fracture) - Griffith's, Irwin-Orowan's, breakdown-zone and finite-thickness model. A linear slip-weakening friction law is briefly presented in relation to the breakdown-zone model as the simplest possible constitutive law for the faulting surface. A brief characterization of the finite-thickness model is accompanied by a short dictionary of basic geologic terms given in the appendix. Two chapters are devoted to the energetic considerations. The first law of thermodynamics is applied to a volume of the elastic continuum in three distinct cases - to a smooth volume without fault (fracture) surface, volume intersecting fracture surface, and volume intersecting fracture surface and containing a fracture edge. The first case leads to the local form of the energy conservation, the second to the energy budget at a point of the fracture surface, the third to the energy budget at a point of the fracture edge. Seismic energy is then defined and all relations given in a concise form in the book by Kostrov and Das (1988) are derived in great detail. A wrong sign in one important relation (4.4.21) in the book is found as well as inconsistent choice of the normal to the surface of the auxiliary volume surrounding the fracture edge (compared to that in the application of the first law of thermodynamics). Because in our future work we will focus on physics of the finite-thickness fault model, the last chapter is a very brief presentation of the most important potential mechanisms of the dynamic fault weakening - strongly related just to the finite-thickness fault model - flash heating and weakening of micro-asperity contacts, thermal pressurization, formation of silica gel, and frictional melting.

Keywords: Earthquake source dynamics, Earthquake energy budget, Seismic energy, Thermodynamics of earthquakes.

Foreword

The presented thesis can be considered as an introductory text on the physics of the earthquake source. The thesis is focused on basic models of the seismoactive fault and earthquake rupture, thermodynamics (energy balance) of the earthquake rupture, seismic energy, and the most important potential mechanisms of the dynamic fault weakening.

The thesis was written within framework of the research conducted by a team of numerical modeling of seismic wave propagation and earthquake motion in the Division of the Physics of the Earth in the Department of Astronomy, Physics of the Earth, and Meteorology under supervision of Professor Peter Moczo.

The topics of the thesis were selected by the supervisor in view of the ongoing and planned research effort of his team.

The author, based on the study of the fundamental books on the physics of the earthquake source as well as selected recent journal papers, explained and summarized basic concepts of the earthquake source dynamics. The author derived in great detail energy balance relations for three basic distinct cases - for a smooth volume without fault (fracture) surface, volume intersecting fracture surface, and volume intersecting fracture surface and containing a fracture edge. The first case leads to the local form of the energy conservation, the second to the energy budget at a point of the fracture surface, the third to the energy budget at a point of the fracture edge. The author also derived in great detail all relations given in a concise form in the book by Kostrov and Das (1988). The author found a wrong sign in one important relation (4.4.21) in the book as well as inconsistent choice of the normal to the surface of the auxiliary volume surrounding the fracture edge (compared to that in the application of the first law of thermodynamics).

The thesis can serve as a useful detail introductory-and-explanatory text for those who are not familiar with the earthquake source dynamics as well as those who want to focus on the source-dynamics related research. The latter is due to the fact that the derived relations are still a subject of recent active research in relation to recent generalizations of the earthquake source model.

Professor Peter Moczo closely supervised the preparation of the thesis during the whole period. Dr. Jozef Kristek and Mgr. Peter Pazak helped with technical preparation of the thesis. Mgr. Peter Pazak also critically read the manuscript. Dr. Massimo Cocco kindly provided his two recent (so far unpublished) manuscripts and commented several aspects of the fault models.

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1 Introduction

Simple model of an earthquake source. Here we closely follow Moczo et al. (2007). An earthquake fault in many problems may be considered as a surface separating two blocks of heterogeneous elastic or viscoelastic medium. A non-zero initial equilibrium stress on the fault surface is assumed as stress caused by tectonic loading plus residual stress after previous earthquakes on the fault. An earthquake itself may be modeled as spontaneous rupture (fracture) propagation along the fault. The propagating rupture radiates seismic waves. The seismic waves propagate from the fault into the Earth's interior. Inside the rupture displacement and particle-velocity vectors are discontinuous across the fault. At the same time traction vector is continuous across the fault surface.

Let $\vec{n}\,(x_i)$ be a unit normal vector to the fault surface pointing from the '$-$' to '$+$' side of the surface. Then slip (discontinuity in the displacement vector across the fault surface) can be defined as

$$\Delta\vec{u}\,(x_i, t) \quad = \quad \vec{u}^{\,+}(x_i^+, t) \;\; - \;\; \vec{u}^{\,-}(x_i^-, t) \; . \tag{1.1}$$

The time derivative of slip, slip rate (discontinuity in the particle-velocity vector across the fault surface) is defined by

$$\Delta\vec{v}\,(x_i, t) \quad = \quad \vec{v}^{\,+}(x_i^+, t) \;\; - \;\; \vec{v}^{\,-}(x_i^-, t) \; . \tag{1.2}$$

Vector of the total traction on the fault is

$$\vec{T}\,(\vec{n}; x_i, t) \quad = \quad \vec{T}^0(\vec{n}; x_i) \;\; + \;\; \Delta\vec{T}\,(\vec{n}; x_i, t) \; , \tag{1.3}$$

where $\vec{T}^0\,(\vec{n}; x_i)$ is the initial traction and $\Delta\vec{T}\,(\vec{n}; x_i, t)$ traction variation (or perturbation). The traction variation is caused by the propagating rupture. At any point of the rupture the total traction is related to slip through the friction law

$$\vec{T} \quad = \quad \vec{T}^f\,(\Delta\vec{u}, \Delta\vec{v}, \theta) \tag{1.4}$$

where $\vec{T}^f$ is frictional traction and θ represents a set of state variables. Equation (1.4), a fault constitutive law, means that the total dynamic traction on the fault is determined by the friction. Given the initial traction and material parameters of the fault, it is the friction law which controls initialization, propagation and healing (arrest) of the rupture.

Consider further only shear faulting. This means that there is no opening of the fault and no interpenetrating of the fault materials. Define frictional strength or fault friction as

$$S \;=\; \mu_f \; | \; \vec{T}_n | \; , \tag{1.5}$$

where μ_f and $\vec{T}_n$ are coefficient of friction and fault-normal component of traction on the fault, respectively. First, assume a locked fault. If, at a point of the fault surface, the magnitude of the shear traction (that is, traction tangential to the fault surface) is smaller than the frictional strength the fault remains locked and slip rate zero at the point. Should the shear traction exceed the frictional strength, slip occurs. The shear traction then varies following the friction law and eventually falls down to the final value. If the final value is equal to the kinematic (dynamic) frictional level, the model is called Orowan's model. If the final value is larger than the kinematic level, the model is called the undershoot model. If the final value is smaller than the kinematic level, the model is called the overshoot model. The slipping, that is, the relative motion of the fault's faces, is opposed by the friction.

Let subscripts sh and n denote the shear and normal components with respect to the fault surface. The boundary conditions on the fault can be formulated as follows.

Shear faulting:

$$\Delta\vec{u}_n \;=\; 0, \quad \Delta\vec{v}_n \;=\; 0, \quad \Delta\vec{u}_{sh} \;\neq\; 0, \quad \Delta\vec{v}_{sh} \;\neq\; 0 \; . \tag{1.6}$$

Shear traction bounded by the frictional strength:

$$| \; \vec{T}_{sh} \; | \;\; \leq \;\; S \; . \tag{1.7}$$

Colinearity of the shear traction and slip rate:

$$S \; \Delta\vec{v}_{sh} \;-\; \vec{T}_{sh} \, (\vec{n}) \;\; |\Delta\vec{v}_{sh}| \;\; = \;\; 0 \; . \tag{1.8}$$

The fact that the frictional traction opposes the slipping is consistent with the colinearity requirement because we consider vector $\vec{n}$ oriented in the direction from the '$-$' to '$+$' side of the fault and slip as the relative motion of the '$+$' side with respect to the '$-$' side of the fault: both $\vec{T}\,(\vec{n})$ and $\Delta\vec{v}$ are viewed from the same side of the fault. If slip was defined as the relative motion of the '$-$' side with respect to the '$+$' side of the fault, requirement of the antiparallelism with the '$+$' sign in eq. (1.8) would be consistent with the frictional traction opposing the relative motion of the fault faces.

When a rupture front reaches a point of the fault and slip starts at that point, the total dynamic traction varies following the friction law. Before the traction at the point reaches the final level, points of the fault in front of the considered point start slipping. Thus, the process of the traction degradation occurs within a finite zone behind the so-called crack tip. This zone is termed cohesive zone or breakdown zone. The friction law determines processes and phenomena in the cohesive zone.

Following Bizzarri and Cocco (2005), the general expression for the coefficient of friction is,

$$\mu_f \quad = \quad \mu_f \; (\; l, \, |\mathit{\Delta}\vec{v}| \; , \mathit{\Psi}_1, \, ..., \, \mathit{\Psi}_N, \, T, \, H, \, \lambda_c, \, h_m, \, g, \, C_e \;) \quad , \tag{1.9}$$

where l is the slip path length

$$l \; = \; \int_{t_0}^{t} |\mathit{\Delta}\vec{v}|(t') \; \mathrm{d}t' \; , \tag{1.10}$$

$|\mathit{\Delta}\vec{v}|$ is modulus of the slip rate, $\mathit{\Psi}_1$, ..., $\mathit{\Psi}_N$ state variables, T temperature accounting for ductility, plastic flow, rock melting and vaporization, H humidity, λ_c characteristic length of the fault surface accounting for roughness and topography of asperity contacts and possibly responsible for mechanical lubrication, h_m material hardness, g gouge parameter accounting for surface consumption and gouge formation during sliding episodes, C_e chemical environment parameter. In general, fault-normal traction in eq. (1.5) should stand for a time-dependent effective normal traction accounting for a pore fluid pressure (which reduces the normal traction).

Due to its generality, equation (1.9) is a very complicated constitutive relation. It is not trivial to fully account for such a constitutive law. In fact, recent numerical modeling of the earthquake source dynamics considerably simplifies the relation due to its methodological complexity, problems to determine values of the parameters, and computer time and memory requirements. For a detailed discussion see, e.g., Cocco and Bizzarri (2002) and Bizzarri and Cocco (2003).

Concept of fracture and model of fracture. Kostrov and Das (1988) define the model of fracture (on p. 54): The set of assumptions related to the transition of particles of the medium from the continuous state to the broken state has been called the 'model of fracture'.

The fracture of the continuous medium must be understood as fracture of 'so large volume' of material which can be described as continuous (at a given scale of macroscopic description, that is, at a given level of 'macroscopicity'). In other words, the fracture is the formation of discontinuity the size of which exceeds the size of particles chosen as elementary particles of continuum. Formation of cracks of smaller size should not be considered as fracturing of the medium at a given level of macroscopic description. At the same time, formation and propagation of macroscopic fracture occurs by nucleation and coalescence (linking) of the cracks of smaller size (that is, microcracks at a given level of macroscopic scale).

In the simplest model, in an ideally brittle medium, particles (of continuum) pass from the continuous to the fractured state along a curve called the crack edge (crack tip). In a more realistic model, model of non-ideally brittle medium, an intermediate state is assumed: some interaction between the crack faces near the edge exists. This interaction depends on the displacement discontinuity and cannot be reduced to friction. Models suggested by Barenblatt (1959), Leonov and Panasyuk (1959) and Dugdale (1960) together with the cohesive-zone models by Ida (1972) and Palmer and Rice (1973) are examples.

Focus of this thesis. In this thesis we focus on the basic models of the seismoactive fault and earthquake rupture, thermodynamic considerations, and seismic energy. We will derive in detail all relations of the earthquake energy balance that were presented in a concise form by Kostrov and Das (1988). Because in our future we plan to focus on physics of the finite-thickness fault model, we will also include a brief presentation of the possible dynamic-weakening mechanisms.

Note. In addition to the books and journal articles directly referred to in the text of the thesis, the following important monographs provide extensive material relevant to the topic: Broberg (1999), Freund (1998), and Lawn (1993).

2 The Stress-strain Relation and Strain-energy Function

2.1 Natural State with Zero Stress and Strain

Here we follow Kostrov and Das (1988) and Aki and Richards (2002). We derive relation for the final elastic energy density (the first relation on page 151 of Kostrov and Das (1988)).

The natural (i.e. initial or reference) state in the elastic body is usually defined as a state with zero stress and zero strain. Strain is then a relative measure of deformation - change in shape of the particles constituting the elastic body due to acting stress. Assuming small deformations, the constitutive relation for a linear elastic body is given by Hooke's law

$$\sigma_{ij} \quad = \quad c_{ijkl}\, \varepsilon_{kl} \ , \tag{2.1}$$

where σ_{ij} is the stress tensor, c_{ijkl} tensor of elastic coefficients, and ε_{kl} (infinitesimal) strain tensor. Both σ_{ij} and ε_{kl} are measured from the natural state.

The equation of motion is

$$\rho\, \ddot{u}_i \quad = \quad \sigma_{ij},_j \ + f_i \ , \tag{2.2}$$

where ρ is density, u_i displacement vector, and f_i body force vector per unit volume.

The application of the 1st law of thermodynamics yields the local form of the energy conservation law

$$\dot{\mathcal{U}} \quad = \quad \sigma_{ij}\, \dot{\varepsilon}_{ij} \ - \ q_{i},_{i} \ \ , \tag{2.3}$$

where $\mathcal{U}$ is volume density of internal (intrinsic) energy and q_i heat flux vector. (Relation (2.3) will be derived in detail in Chapter 4.)

The strain energy function $\mathcal{W}$ is defined by

$$\sigma_{ij} \quad = \quad \frac{\partial \mathcal{W}}{\partial \varepsilon_{ij}} \ . \tag{2.4}$$

If the strain-energy function exists for a deformation, and $\mathcal{W} = 0$ in the natural state ($\sigma_{ij} = 0,\ \varepsilon_{ij} = 0$), eq. (2.4) combined with Hooke's law yields

$$\mathcal{W} = \frac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij}\ . \tag{2.5}$$

(Note that if we formally assumed $\mathcal{W} = \mathcal{W}^0$ in the natural state, we would get $\mathcal{W} = \mathcal{W}^0 + \frac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij}$. The other relations remain unchanged.)

Using Hooke's law, formal re-indexing ($i \leftrightarrow k,\ j \leftrightarrow l$), and symmetry $c_{ijkl} = c_{klij}$ we obtain

$$\dot{\mathcal{W}} = \sigma_{ij}\,\dot{\varepsilon}_{ij}\ . \tag{2.6}$$

Substitution of eq. (2.6) into eq. (2.3) gives

$$\dot{\mathcal{U}} = \dot{\mathcal{W}} - q_{i,i}\ . \tag{2.7}$$

If the deformation is adiabatic, that is, if $q_i = 0$, we can choose

$$\mathcal{W} = \mathcal{U}\,. \tag{2.8}$$

2.2 Natural State with Non-zero Stress (Prestress) and Zero Strain

The concept of the natural state described in the previous section is not applicable in the theory of the earthquake source because the Earth's interior is under considerable stress - at least hydrostatic. In general, the state just before an earthquake is characterized by initial stress σ^0_{ij} and initial strain ε^0_{ij}. The initial strain is not small and the stress - strain relation is nonlinear.

In order to avoid a nonlinear treatment, the state just before the earthquake is defined as the initial state with non-zero initial stress σ^0_{ij} (also called prestress) but zero initial strain.

The total stress during the earthquake, σ_{ij}, is then

$$\sigma_{ij} = \sigma^0_{ij} + \tau_{ij}\ , \tag{2.9}$$

where τ_{ij} is the stress variation (perturbation) due to the earthquake. Assuming the linear theory of elasticity with initial shear stress components small compared to the elastic moduli, and u_i and ε_{ij} measured from the initial state before the earthquake,

$$\tau_{ij} \;=\; c_{ijkl}\,\varepsilon_{kl} \tag{2.10}$$

and

$$\sigma_{ij} \;=\; \sigma^0_{ij} \;+\; c_{ijkl}\,\varepsilon_{kl}\;. \tag{2.11}$$

The density of the strain energy is then

$$\mathcal{W} \;=\; \sigma^0_{ij}\,\varepsilon_{ij} \;+\; \frac{1}{2}\,\tau_{ij}\,\varepsilon_{ij}\;. \tag{2.12}$$

Or, considering relation (2.9),

$$\mathcal{W} \;=\; \frac{1}{2}\,\left(\sigma^0_{ij} \;+\; \sigma_{ij}\right)\,\varepsilon_{ij}\;. \tag{2.13}$$

It is easy to check, using $\dfrac{\partial\,\varepsilon_{kl}}{\partial\,\varepsilon_{ij}} \;=\; \delta_{ik}\,\delta_{lj}$, that

$$\frac{\partial\,\mathcal{W}}{\partial\,\varepsilon_{ij}} \;=\; \sigma_{ij} \tag{2.14}$$

follows from relation (2.12) or (2.13).

The time differentiation of relation (2.13) gives, using eqs. (2.9) and (2.10), and symmetry $c_{klij} \;=\; c_{ijkl}$,

$$\dot{\mathcal{W}} \;=\; \sigma_{ij}\,\dot{\varepsilon}_{ij}\;. \tag{2.15}$$

Due to symmetry of the stress tensor, $\sigma_{ij} \;=\; \sigma_{ji}$, relation (2.15) can be written as

$$\dot{\mathcal{W}} \;=\; \sigma_{ij}\,\dot{u}_{i,j}\;. \tag{2.16}$$

Let us integrate relation (2.16) from the initial time t_0 until the final time t_1 (after the earthquake) realizing that $u_i\,(t_0) \;=\; 0$ and $\tau_{ij}\,(t_0) \;=\; 0$. For brevity we will use the following notation:

$$\mathcal{W}^0 \;=\; \mathcal{W}\,(t_0) \;\;,\;\; \mathcal{W}^1 \;=\; \mathcal{W}\,(t_1) \;\;,\;\; u^1_i \;=\; u_i\,(t_1) \;\;,\;\; \tau^1_{ij} \;=\; \tau_{ij}\,(t_1)\;. \tag{2.17}$$

The integration of relation (2.16) gives

$$
\begin{aligned}
\mathcal{W}^1 \; - \; \mathcal{W}^0 \; &= \int_{t_0}^{t_1} \sigma_{ij} \, \dot{u}_{i,j} \; \mathrm{d}t \; = \\
&= \int_{t_0}^{t_1} \left(\sigma^0_{ij} \; + \; \tau_{ij} \right) \; \dot{u}_{i,j} \; \mathrm{d}t \; = \\
&= \sigma^0_{ij} \, u^1_{i,j} \; + \; \int_{t_0}^{t_1} \tau_{ij} \, \dot{u}_{i,j} \; \mathrm{d}t \; = \\
&= \sigma^0_{ij} \, u^1_{i,j} \; + \; [\, \tau_{ij} \, u_{i,j} \,]^{t=t_1}_{t=t_0} \; - \; \int_{t_0}^{t_1} \dot{\tau}_{ij} \, u_{i,j} \; \mathrm{d}t \; = \\
&= \sigma^0_{ij} \, u^1_{i,j} \; + \tau^1_{ij} \, u^1_{i,j} \; - \; \int_{t_0}^{t_1} \dot{\tau}_{ij} \, u_{i,j} \; \mathrm{d}t \; .
\end{aligned}
\tag{2.18}
$$

The integrand in the last line of eq. (2.18) can be written as

$$
\begin{aligned}
\dot{\tau}_{ij} \, u_{i,j} \quad &= \quad c_{ijkl} \, \dot{u}_{k,l} \;\; u_{i,j} \;\; = \\
&= \quad c_{ijkl} \, \frac{\mathrm{d}}{\mathrm{d}t} \left(\, u_{k,l} \;\; u_{i,j} \, \right) \;\; - \;\; c_{ijkl} \, u_{k,l} \;\; \dot{u}_{i,j} \;\; .
\end{aligned}
\tag{2.19}
$$

Using formal re-indexing ($i \;\leftrightarrow\; k$, $j \;\leftrightarrow\; l$), and symmetry $c_{klij} \;=\; c_{ijkl}$ we obtain

$$
c_{ijkl} \, \dot{u}_{k,l} \;\; u_{i,j} \;\; = \;\; c_{ijkl} \, \frac{\mathrm{d}}{\mathrm{d}t} \left(\, u_{k,l} \;\; u_{i,j} \, \right) \;\; - \;\; c_{ijkl} \, \dot{u}_{k,l} \;\; u_{i,j}
\tag{2.20}
$$

and

$$
c_{ijkl} \, \dot{u}_{k,l} \;\; u_{i,j} \;\; = \;\; \frac{1}{2} \, c_{ijkl} \, \frac{\mathrm{d}}{\mathrm{d}t} \left(\, u_{k,l} \;\; u_{i,j} \, \right) \, .
\tag{2.21}
$$

Using eqs. (2.19) and (2.21) the integral in the last line of eq. (2.18) can be written as

$$
- \int_{t_0}^{t_1} \frac{1}{2} \, c_{ijkl} \, \frac{\mathrm{d}}{\mathrm{d}t} \left(\, u_{k,l} \;\; u_{i,j} \, \right) \mathrm{d}t \;\; = \;\; - \, \frac{1}{2} \, c_{ijkl} \, u^1_{k,l} \;\; u^1_{i,j} \;\; = \;\; - \, \frac{1}{2} \, \tau^1_{ij} \, u^1_{i,j} \;\; .
\tag{2.22}
$$

Then eq. (2.18) yields

$$
\mathcal{W}^1 \; - \; \mathcal{W}^0 \;\; = \;\; \sigma^0_{ij} \, u^1_{i,j} \; + \tau^1_{ij} \, u^1_{i,j} \; - \; \frac{1}{2} \, \tau^1_{ij} \, u^1_{i,j}
\tag{2.23}
$$

and

$$
\mathcal{W}^1 \;\; = \;\; \mathcal{W}^0 \; + \; \sigma^0_{ij} \, u^1_{i,j} \; + \; \frac{1}{2} \, \tau^1_{ij} \, u^1_{i,j}
\tag{2.24}
$$

or, using relation (2.9) ,

$$
\mathcal{W}^1 \;\; = \;\; \mathcal{W}^0 \; + \; \frac{1}{2} \; \left(\, \sigma^1_{ij} \; + \; \sigma^0_{ij} \, \right) \, u^1_{i,j} \;\; ,
\tag{2.25}
$$

which are the desired relations given in Kostrov and Das (1988) on page 151.

The medium is in equilibrium in the initial state:

$$0 \quad = \quad \sigma^0_{ij},_j \; + f_i \; . \tag{2.26}$$

Because even in the large earthquakes displacements do not exceed a few meters, it is reasonable to consider that the gravitational force f_i does not change during the earthquake. The equation of motion during the earthquake is

$$\rho \, \ddot{u}_i \quad = \quad \sigma_{ij},_j \; + \; f_i \; , \tag{2.27}$$

that is

$$\rho \, \ddot{u}_i \quad = \quad \left(\, \sigma^0_{ij} \; + \; \tau_{ij} \, \right),_j \; + \; f_i \; . \tag{2.28}$$

Subtracting equation of equilibrium before the earthquake, eq. (2.26), from eq. (2.28) gives

$$\rho \, \ddot{u}_i \quad = \quad \tau_{ij},_j \tag{2.29}$$

during the earthquake.

3 Models of Seismoactive Fault and Earthquake Rupture

Here we introduce basic models of fracture as well as the realistic model of a seismoactive fault and earthquake fracturing. What follows is based on texts by Kostrov and Das (1988), Scholz (2002), Cocco et al. (2007), Rice and Cocco (2007), and Moczo et al. (2007).

3.1 Griffith's Static Crack

A strength of a brittle solid material can be defined as the maximum stress that the given material is capable to support under given conditions. A fracture, a loss of continuity of the material, must involve the breaking of atomic bonds across a lattice plane. An estimate of the stress required to break the atomic bonds gives a theoretical strength of the material. The theoretical strength from such estimate is 5-10 GPa. This value is, however, two to three orders of magnitude greater than the strength of real materials.

Real materials contain imperfections that cause stress concentrations - locally increased stresses. These stress concentrations can lead to material failures at much lower stresses than the theoretical strength of materials.

Consider, as a simple example, an elliptical hole in an elastic plate under a uniform tensile stress σ_L. The stress concentration at the ends of the elliptical hole can be approximated as

$$\sigma \approx \sigma_L \left(1 + 2 \frac{c}{b} \right) , \tag{3.1}$$

where c and b are semiaxes, and $c > b$. For a given fixed value of σ_L, the stress at the ends of the elliptical hole increases proportionally to c/b. Therefore, for a long narrow elliptical hole, that is, crack, the stress at the ends (at the crack tips) can reach value of the theoretical strength even for the loading stress σ_L much smaller than the theoretical strength. Consequently, an increase of the stress concentration can lead to a dynamic instability.

Griffith (1920, 1924) formulated the problem of the tensile crack in the form of the energy balance, that is, he applied the 1st law of thermodynamics to a volume of an elastic continuum containing a crack. Griffith's formulation can be written in the form

$$\delta (A + Q) = \delta \Pi + 2 \gamma \, \delta \Sigma . \tag{3.2}$$

The work of external forces (tractions acting at a surface enclosing the volume plus body forces acting throughout the volume) and the heat supplied to the volume, $\delta (A + Q)$, is equal to the change of the kinetic and potential energies plus energy dissipation, $\delta \Pi$, and an amount of energy necessary to create fracture surface, $2 \gamma \, \delta \Sigma$. Here, γ is the specific surface energy necessary to create a unit area of fracture surface, $\delta \Sigma$ is the fracture surface increment, and factor 2 accounts for two faces of the fracture surface. Griffith considered γ to be a material constant (this is, in fact, not so; γ may depend on the velocity of the fracture propagation, temperature and other thermodynamic parameters). Quantity 2γ is called specific fracture work or crack-driving force. Equation (3.2) can be viewed as Griffith's fracture criterion. Although the physical meaning of the criterion is clear, eq. (3.2) as a global criterion is not readily applicable for practical applications.

Griffith referred his analysis to the submolecular level. In this sense the Griffith's model is the basic model of the microcrack. The fracture surface behind the tensile crack tip is cohesionless. The fracture is a balance between the available energy to drive the crack and energy absorbed by inelastic processes exclusively at the crack tip. Because a finite amount of energy is spent at the crack tip, Griffith's crack has stress singularity at the crack tip. Given these characteristics, it is clear that Griffith's crack cannot serve as approximation to the real earthquake fracturing.

3.2 Irwin-Orowan's Crack

Irwin's contribution to the theory of fracture (Irwin, 1948, 1960) included the application of the stress intensity factors, and, together with Orowan (1952), extension of the Griffith's concept to microcracks in steel and concept of a quasi-brittle fracture.

Irwin characterized the stress at and ahead of the crack tip using a stress intensity factors. Irwin's fracture criterion requires that the stress intensity factor be equal to the critical stress intensity factor. For simple planar cohesionless/frictionless cracks he found simple analytical expressions for the stress and critical stress intensity factors. For the tensile crack, the fracture criterion is (that is, the condition for crack propagation is met if)

$$G_c = \frac{K_c^2}{E} = 2\gamma. \tag{3.3}$$

Here G_c is the fracture energy, K_c is the critical stress intensity factor, E is the effective Young modulus, and γ is the specific surface energy. The criterion yields fracture energy as the available energy for driving crack. The energy is absorbed by inelastic processes at the crack tip. It follows for the Griffith's crack that all the fracture energy is surface energy.

Irwin extended the Griffith's concept to microcracks in steel and proposed the concept of quasi-brittle fracture. In fact, his extension was methodologically an extension to any higher macroscopic level. Around the fracture edge, at a given macroscopic level, in a so-called process zone, complex microcracking and plastic deformation occurs, and microcracks link to create the macroscopic fracture (macrocrack). The concept of the brittle fracture can be still applied by introducing the effective surface energy γ_{eff} which includes all the energy losses during fracturing and, particularly, plastic work. In other words, the effective surface energy is a macroscopic measure of the total energy absorbed during the fracture development within the process zone at the crack tip. Consequently, the fracture energy in Irwin's (or Irwin-Orowan's) crack model is

$$G_c = 2\gamma_{eff}. \tag{3.4}$$

We can point out again that although γ_{eff} is called surface energy - in correspondence to the surface energy appearing in eq. (3.3) - it includes not only the surface energy but also other dissipative mechanisms such as heat.

Note that stress has singularity at the crack tip in both Griffith's and Irwin's crack models.

Although Griffith, Irwin and Orowan considered only tensile crack, the displacement field of cracks can be categorized into three distinct modes: Mode I (tensile or opening), Mode II (shear in-plane) and Mode III (shear anti-plane). In the tensile Mode I crack-wall displacements are normal to the plane of the crack. In the shear in-plane Mode II crack-wall displacements are in the plane of the crack and normal to the crack edge. In the shear anti-plane Mode III crack-wall displacements are in the plane of the crack and parallel to the crack edge. The three Modes are illustrated in Fig. 1.

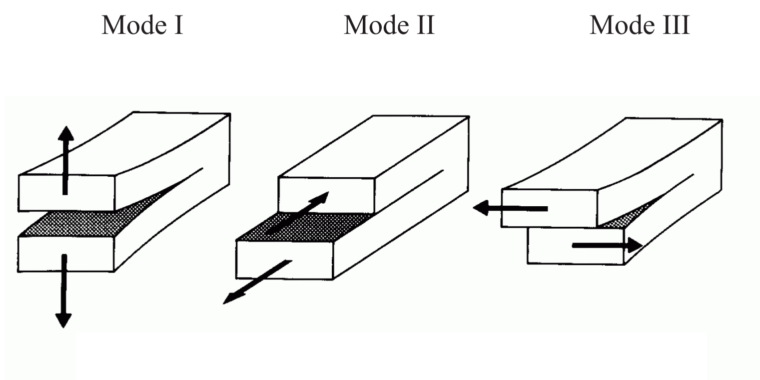

Figure 1. Three distinct modes of 2D cracks. Here 2D means that planes perpendicular to the plane of crack and crack edge are equivalent. According to Scholz (2002).

3.3 Breakdown Zone Model

Cohesionless tensile crack and frictionless shear cracks are too far from being realistic. Therefore Barenblatt (1959) for tensile crack and Ida (1972) for shear cracks introduced the concept of breakdown process in which finite force between two crack faces varies continuously from an initial level to some minimum level as slip between two crack faces increases. The existence of the breakdown zone means finite stress at the crack tip.

One of the main aspects relevant for earthquake rupturing and particularly for the fault weakening during earthquake is the traction evolution. The traction evolution from the initial value to the final (or minimum) value including the phase of traction drop with an increasing slip occurs within the breakdown (or process) zone that is a finite-extent zone at the crack edge. Different physical processes can yield a traction evolution consistent with that behavior. The shapes of the traction-slip curves can differ for different constitutive formulations for the breakdown process but all of them have to be consistent with the well established traction degradation with an increasing slip.

Cocco et al. (2007) defined the breakdown zone as the region of a locally 2D crack between the crack tip and point having minimum traction behind the crack tip.

One formulation of the breakdown zone is the linear slip weakening model (Ida, 1972; Palmer and Rice, 1973; Andrews, 1976a,b). Value of the coefficient of friction in the linear slip weakening friction law decreases linearly from the value of the coefficient of static friction, μ_s, down to the value of the coefficient of kinematic (also called dynamic) friction, μ_d, over a characteristic (also called critical) distance D_c :

$$\begin{aligned} \mu_f &= \mu_s - \frac{\mu_s - \mu_d}{D_c}\, l \qquad ; \quad l < D_c \,, \\ \mu_f &= \mu_d \qquad\qquad\qquad ; \quad l \geq D_c \,, \end{aligned} \tag{3.5}$$

where l is a slip path length defined by relation (1.10). Equivalently, the slip weakening friction law can be expressed in terms of the corresponding shear tractions:

$$\begin{aligned} |\vec{T}^f_{sh}| &= |\vec{T}^s_{sh}| - \frac{|\vec{T}^s_{sh}| - |\vec{T}^d_{sh}|}{D_c}\, l \quad ; \quad l < D_c \,, \\ |\vec{T}^f_{sh}| &= |\vec{T}^d_{sh}| \qquad\qquad\qquad ; \quad l \geq D_c \,. \end{aligned} \tag{3.6}$$

Here $|\vec{T}^s_{sh}|$ and $|\vec{T}^d_{sh}|$ are the static (also called yield) and kinematic frictional shear tractions, respectively. In other words, the frictional strength depends only on a cumulative slip path length. Considering the slip weakening friction law means that the evolution of the traction on the fault is 'prescribed' a priori. Though the slip weakening friction law is relatively very simple, in practical applications it is, in fact, very difficult to estimate or determine reasonable values of coefficients of the static and kinematic frictions, and value of the critical distance.

Because the slip weakening model involves frictional sliding (that occurs everywhere behind the crack tip), mechanical work done against frictional stress is irreversible.

3.4 Geological (Finite-thickness) Fault-zone Model

Field geological observations clearly confirm that seismoactive faults or, better, fault zones have finite thickness and relatively complex structure. Rice and Cocco (2007) summarize results of field geological and seismological investigations and present a model of a major fault zone. Their model is shown in Fig. 2.

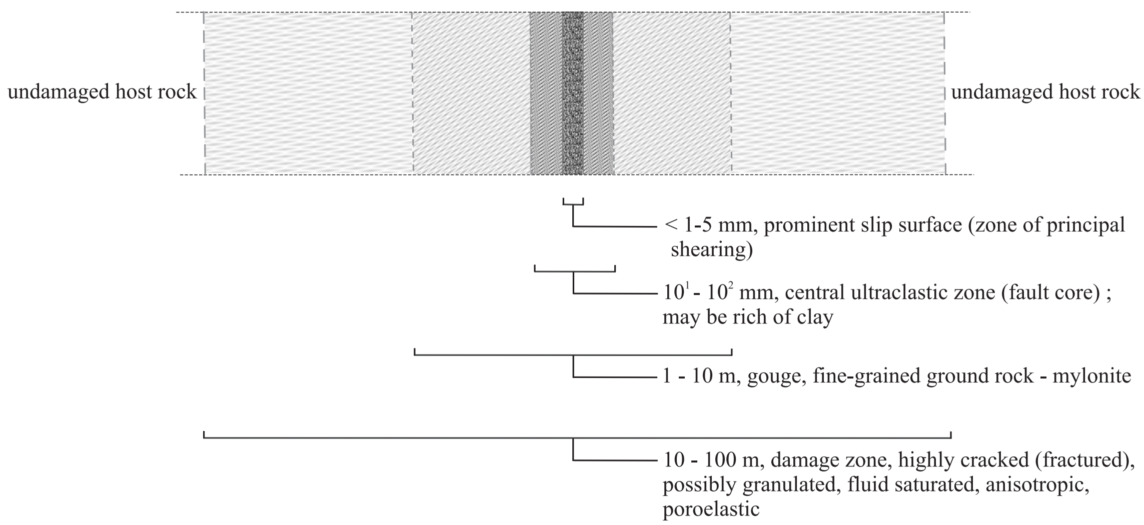

Figure 2. Schematic picture of a model of a major fault zone based on Rice and Cocco (2007) and Chester et al. (1993).

A prominent slip surface (thin zone) which may be less than 1-5 mm thick is at the center of the fault structure. It is a thin zone of principal shearing. The prominent slip surface is inside a central ultracataclastic zone (fault core) which may be 10s-100s mm thick and rich of clay. (Basic explanations of relevant geological terms is given in the Appendix.) The central ultracataclastic zone is surrounded by a gouge or foliated gouge which may be 1-10 m thick. The gouge layer consist of fine-grained ground rock (mylonite). Next to the gouge layer is a broad (10s-100s m thick) damage zone. The material in the damage zone is highly cracked (fractured), possibly granulated, poroelastic, fluid saturated, and anisotropic.

The picture of the fault zone should be viewed as schematic. The text in Rice and Cocco (2007) on the structure seems a little bit ambiguous. Although Fig. 2 clearly distinguishes the above mentioned zones and layers, the text itself, in our opinion, mixes up terms fault core, gouge and damage zone. For example, the text says 'In general, the principal slipping zone contains wear materials or gouge, which can be cohesive or incohesive.' At other place it is said 'The observations ... allow the proposition of the fault zone model characterized by the presence of localized slip in a thin zone, the presence of frictional wear or gouge, a fault core composed of cataclasite and ultracataclasite, and a broader damage zone (highly fractured, anisotropic, and poroelastic).'

Personal communication with Massimo Cocco clarified that different authors use term 'gouge' in different ways. Many of them consider 'gouge' material as cracked, fluid saturated and poroelastic material. They assume that such a material fills the fault core. M. Cocco assumes that the material inside the fault core is much more cracked and fluid saturated than material in the gouge and damage zones shown in Fig. 2.

In Chapter 6 we will briefly present candidate fault-weakening mechanisms and processes which may partly clarify some aspects of present understanding of the fault-zone structure.

Cocco et al. (2007) point out the main implication of the geological field observations: faults have a finite thickness (although sometimes very narrow, $\approx$ mm) and are filled by gouge and wear materials produced during faulting. One significant consequence of the complex fault structure is the necessity to consider physical quantities characterizing the earthquake rupture process (shear traction, slip and slip rate) in a macroscopic sense or as a phenomenological description of complex processes. The shear traction, slip and slip rate should be considered as equivalent physical quantities acting on the walls of the fault zone of the finite thickness.

Cocco et al. (2007) present application and generalization of the concepts and analyses developed for a fault surface to the fault model with finite thickness. In order to understand their generalization it is necessary to fully understand concepts of the energy balance of the faulting surface.

4 Thermodynamics of Earthquake Rupture - Earthquake Energy Balance

Here we closely follow Kostrov and Das (1988) - we perform a detailed derivation of relations presented in a very concise style by Kostrov and Das. We also show misprints (errors?) and inconsistencies in the text by Kostrov and Das. We consider it very important for our future work in the topic to perform a detailed derivation of all formulas.

The concepts of seismic energy and fracture energy are still being investigated. They are closely related to the problem of seismic efficiency - how much of the total released energy goes into the energy of radiated seismic waves.

Recent discussion and investigation is well reflected by, e.g., Cocco et al. (2007), Fukuyama (2005) and Kanamori (2001).

4.1 Energy Conservation Law for a Continuous Medium

Consider a material volume V of continuum with surface S in which material parameters are continuous. Let $\vec{n}$ be a normal vector to surface S pointing from interior of volume V outward. Consider body force $\vec{f}\,(x_k, t)$ acting in volume V, traction $\vec{T}\,(\vec{n})$ acting at surface S, and heat-flux vector $\vec{q}\,(\vec{n})$. Here x_k ; $k \in 1, 2, 3$ are Cartesian coordinates and t is time. The configuration is shown in Fig. 3. Given the chosen configuration, $q_i n_i$ is the rate at which heat is transmitted (per unit area) in the $\vec{n}$

direction across surface locally perpendicular to $\vec{n}$. $\vec{T}\,(\vec{n})$ is the traction with which material outside volume V acts upon material inside V.

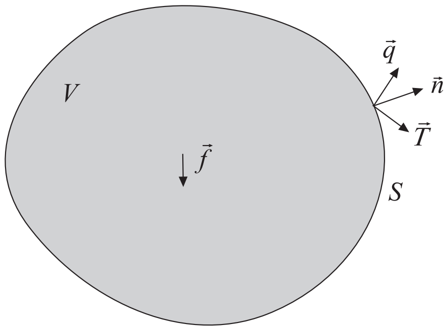

Figure 3. Material volume V of a smooth continuum bounded by surface S.

The 1st law of thermodynamics, the energy conservation law, applied to volume V means

$$\dot{A} \;+\; \dot{Q} \;=\; \dot{U} \;+\; \dot{K}\,, \tag{4.1}$$

where $\dot{A}$ is the rate of work of external forces (body forces and surface tractions), $\dot{Q}$ is the rate of heat supplied to volume V, $\dot{U}$ is the rate of increase of internal energy, and $\dot{K}$ is the rate of increase of kinetic energy. Elaborate individual terms in eq. (4.1).

The rate of doing mechanical work is

$$\begin{aligned}
\dot{A} &= \int_S T_i\,\dot{u}_i\,\mathrm{d}S + \int_V f_i\,\dot{u}_i\,\mathrm{d}V = \\
&= \int_S \sigma_{ij}\,n_j\,\dot{u}_i\,\mathrm{d}S + \int_V f_i\,\dot{u}_i\,\mathrm{d}V = \\
&= \int_V [\,(\,\sigma_{ij}\,\dot{u}_i\,)_{,j} + f_i\,\dot{u}_i\,]\,\mathrm{d}V = \\
&= \int_V [\,(\,\sigma_{ij,j} + f_i\,)\,\dot{u}_i + \sigma_{ij}\,\dot{u}_{i,j}\,]\,\mathrm{d}V = \\
&= \int_V [\,\rho\,\ddot{u}_i\,\dot{u}_i + \sigma_{ij}\,\dot{u}_{i,j}\,]\,\mathrm{d}V = \\
&= \int_V \left[\frac{1}{2}\,\rho\,\frac{\partial}{\partial t}\,(\dot{u}_i\,\dot{u}_i) + \sigma_{ij}\,\dot{u}_{i,j}\right]\mathrm{d}V\,,
\end{aligned} \tag{4.2}$$

where we applied Gauss's divergence theorem and used equation of motion,

$$\rho\,\ddot{u}_i \;=\; \sigma_{ij,j} \;+\; f_i\,. \tag{4.3}$$

Note that, obviously, we would get the final relation for the rate of mechanical work even if $f_i = 0$. The rate of heating is

$$\dot{Q} = - \int_S q_i \, n_i \, \mathrm{d}S = - \int_V q_{i,i} \, \mathrm{d}V \, . \tag{4.4}$$

Denoting a volume density of internal energy by $\mathcal{U}$, the rate of increase of internal energy can be written as

$$\dot{U} = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \mathcal{U} \, \mathrm{d}V = \int_V \dot{\mathcal{U}} \, \mathrm{d}V \, . \tag{4.5}$$

The time rate of increase of the kinetic energy can be written as

$$\dot{K} = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{1}{2} \rho \, \dot{u}_i \, \dot{u}_i \, \mathrm{d}V = \int_V \frac{1}{2} \rho \frac{\partial}{\partial t} \left(\dot{u}_i \, \dot{u}_i \right) \mathrm{d}V \, . \tag{4.6}$$

In eqs. (4.5) and (4.6) we made use of the fact that in the Lagrangian description the volume moves with the particles and the particle mass $\rho \, \mathrm{d}V$ is constant in time. Substitution of eqs. (4.2), (4.4) - (4.6) into eq. (4.1) yields

$$\int_V \left(\sigma_{ij} \, \dot{u}_{i,j} - q_{i,i} - \dot{\mathcal{U}} \right) \mathrm{d}V = 0 \, . \tag{4.7}$$

Due to arbitrariness of volume V and continuity of the integrand, the integrand must vanish identically. Then

$$\dot{\mathcal{U}} = \sigma_{ij} \, \dot{u}_{i,j} - q_{i,i} \, . \tag{4.8}$$

Because, due to symmetry of the stress tensor,

$$\sigma_{ij} \, \dot{u}_{i,j} = \sigma_{ij} \, \dot{\varepsilon}_{i,j} \, , \tag{4.9}$$

eq. (4.8) can be written as

$$\dot{\mathcal{U}} = \sigma_{ij} \, \dot{\varepsilon}_{i,j} - q_{i,i} \, . \tag{4.10}$$

Consider time derivative of the function of the energy of deformation $\mathcal{W}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} c_{ijkl} \, \varepsilon_{kl} \, \varepsilon_{ij} \right) &= \frac{1}{2} c_{ijkl} \, \dot{\varepsilon}_{kl} \, \varepsilon_{ij} + \frac{1}{2} c_{ijkl} \, \varepsilon_{kl} \, \dot{\varepsilon}_{ij} = \\ &= \frac{1}{2} c_{klij} \, \dot{\varepsilon}_{ij} \, \varepsilon_{kl} + \frac{1}{2} c_{ijkl} \, \varepsilon_{kl} \, \dot{\varepsilon}_{ij} = \\ &= \frac{1}{2} c_{ijkl} \, \dot{\varepsilon}_{ij} \, \varepsilon_{kl} + \frac{1}{2} c_{ijkl} \, \varepsilon_{kl} \, \dot{\varepsilon}_{ij} = \\ &= c_{ijkl} \, \varepsilon_{kl} \, \dot{\varepsilon}_{ij} \, , \end{aligned} \tag{4.11}$$

where we used symmetry of tensor of elastic coefficients,

$$c_{klij} \quad = \quad c_{ijkl} \ . \tag{4.12}$$

Thus we have

$$\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}t} \quad = \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij}\right) \quad = \quad \sigma_{ij}\,\dot{\varepsilon}_{ij} \ . \tag{4.13}$$

Equations (4.8) and (4.10) are local forms of the energy conservation law. They mean that the rate of increase of the internal energy is equal to the rate of energy of deformation (the rate of internal work) plus the rate of heating.

4.2 Energy Budget on a Fault Surface

Consider volume V intersecting fracture surface $\Sigma(t)$ over an area $\Delta\Sigma$ not containing fracture's edge. The geometrical configuration of the problem is illustrated in Fig. 4. At the fracture surface, displacement vector is discontinuous. The displacement

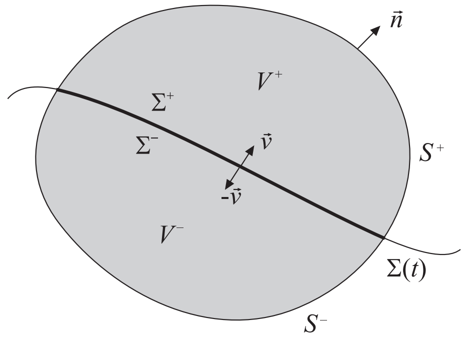

Figure 4. Material volume V intersecting fracture surface $\Sigma(t)$. The volume does not contain fracture's edge.

discontinuity (slip) is defined by

$$\Delta u_i \quad = \quad u_i\left(\Sigma^+\right) \ - \ u_i\left(\Sigma^-\right) \quad = \quad u_i^+ \ - \ u_i^- \ . \tag{4.14}$$

Due to friction we have to assume that part of mechanical work against frictional traction produces heat at the fracture surface. The heat flux from the fracture surface causes discontinuity

$$\Delta\, q_i \quad = \quad q_i^+ \ - \ q_i^- \ . \tag{4.15}$$

Let γ be a specific internal surface energy (the energy needed to create a unit area of one face of a fracture). Then the rate of energy spent in creation area $\Delta\Sigma$ of the fracture surface is

$$\dot{U}_\Sigma = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Delta\Sigma} 2\,\gamma\,\mathrm{d}S = \int_{\Delta\Sigma} 2\,\dot{\gamma}\,\mathrm{d}S\,. \tag{4.16}$$

The 1st law of thermodynamics applied to volume V can be written as

$$\dot{A} + \dot{Q} = \dot{U} + \dot{K} + \dot{U}_\Sigma\,. \tag{4.17}$$

The heat produced by the fracture is included in $\dot{Q}$. In principle, $\dot{U}_\Sigma$ could be included in $\dot{U}$. $\dot{U}$ in eq. (4.17) relates only to the internal energy in volume, say, $\dot{U}_V$, not to the internal energy of the fracture surface. In this respect, eq. (4.17) can be written as

$$\dot{A} + \dot{Q} = \dot{U}_V + \dot{K} + \dot{U}_\Sigma\,. \tag{4.18}$$

The rate of doing mechanical work is

$$\begin{aligned}\dot{A} &= \int_S T_i\,\dot{u}_i\,\mathrm{d}S + \int_V f_i\,\dot{u}_i\,\mathrm{d}V = \\ &= \int_S \sigma_{ij}\,n_j\,\dot{u}_i\,\mathrm{d}S + \int_V f_i\,\dot{u}_i\,\mathrm{d}V\,.\end{aligned} \tag{4.19}$$

Because displacement is discontinuous across surface Σ, Gauss's divergence theorem cannot be directly applied to volume V with surface S. Therefore we consider

$$\begin{aligned}\int_V (\sigma_{ij}\,\dot{u}_i)_{,j}\,\mathrm{d}V &= \int_{V_+} (\sigma_{ij}^+\,\dot{u}_i^+)_{,j}\,\mathrm{d}V + \int_{V_-} (\sigma_{ij}^-\,\dot{u}_i^-)_{,j}\,\mathrm{d}V = \\ &= \int_{S_+} \sigma_{ij}^+\,\dot{u}_i^+\,n_j\,\mathrm{d}S + \int_{\Delta\Sigma} \sigma_{ij}^+\,\dot{u}_i^+\,(-\,\nu_j)\,\mathrm{d}S + \\ &+ \int_{S_-} \sigma_{ij}^-\,\dot{u}_i^-\,n_j\,\mathrm{d}S + \int_{\Delta\Sigma} \sigma_{ij}^-\,\dot{u}_i^-\,\nu_j\,\mathrm{d}S = \\ &= \int_S \sigma_{ij}\,\dot{u}_i\,n_j\,\mathrm{d}S - \int_{\Delta\Sigma} \sigma_{ij}\,\Delta\dot{u}_i\,\nu_j\,\mathrm{d}S\,,\end{aligned} \tag{4.20}$$

since traction is continuous across the fault surface $\Delta\Sigma$. Substituting eq. (4.20) into eq. (4.19) we obtain

$$\dot{A} = \int_V [\,(\sigma_{ij}\,\dot{u}_i)_{,j} + f_i\,\dot{u}_i\,]\,\mathrm{d}V + \int_{\Delta\Sigma} \sigma_{ij}\,\Delta\dot{u}_i\,\nu_j\,\mathrm{d}S\,. \tag{4.21}$$

The first integral on the l.h.s. of the equation can be rewritten in the same way as in section 4.1,

$$\dot{A} \;=\; \int_V \left[\; \frac{1}{2}\,\rho\,\frac{\partial}{\partial t}\;(\,\dot{u}_i\,\dot{u}_i) \;+\; \sigma_{ij}\,\dot{\varepsilon}_{ij}\;\right]\,\mathrm{d}V \;+\; \int_{\Delta\Sigma} \sigma_{ij}\,\Delta\dot{u}_i\,\nu_j\,\mathrm{d}S\;, \qquad (4.22)$$

where we also used eq. (4.9). Accounting in a similar way for the discontinuity in the heat flux at fracture surface $\Delta\Sigma$ we obtain for the rate of heating

$$\dot{Q} \;=\; -\int_S q_i\,n_i\,\mathrm{d}S \;=\; -\int_V q_{i,i}\;\mathrm{d}V \;-\; \int_{\Delta\Sigma} \Delta q_i\,\nu_i\,\mathrm{d}S\;. \qquad (4.23)$$

The rates of change of internal and kinetic energies are

$$\dot{U}_V \;=\; \int_V \dot{\mathcal{U}}\,\mathrm{d}V \qquad (4.24)$$

and

$$\dot{K} \;=\; \int_V \frac{1}{2}\,\rho\,\frac{\partial}{\partial t}\,(\,\dot{u}_i\,\dot{u}_i\,)\;\mathrm{d}V\;. \qquad (4.25)$$

Substitution of eqs. (4.16) and (4.22) - (4.25) into eq. (4.18) yields

$$\int_V \left(\,\sigma_{ij}\,\dot{\varepsilon}_{ij} \;-\; q_{i,i} \;-\dot{\mathcal{U}}\right)\,\mathrm{d}V \;=$$

$$=\; \int_{\Delta\Sigma} [\,(\,-\sigma_{ij}\,\Delta\dot{u}_i \;+\; \Delta q_j\,)\,\nu_j \;+\; 2\dot{\gamma}\,]\;\mathrm{d}S\;. \qquad (4.26)$$

The l.h.s. of eq. (4.26) vanishes due to eq. (4.10). Then, due to arbitrariness of $\Delta\Sigma$ and continuity of the integrand we obtain

$$\sigma_{ij}\,\nu_j\,\Delta\dot{u}_i \;=\; 2\dot{\gamma} \;+\; \Delta q_i\,\nu_i\;. \qquad (4.27)$$

The equation means that the frictional work is partly spent in the change of the internal surface energy and partly released into the medium as heat.

Comment on the surface energy:
Assume that the surface energy depends only on the thermodynamic state of the fracture itself and not on the relative motion of its faces. Let, e.g., be the thermodynamic state determined by the surface temperature,

$$\gamma \;=\; \gamma\,(\,T\,)\;. \qquad (4.28)$$

Then eq. (4.27) can be written as

$$\sigma_{ij}\,\nu_j\,\Delta\dot{u}_i \;=\; 2\,\frac{\partial\gamma}{\partial T}\,\dot{T} \;+\; \Delta q_i\,\nu_i\;. \qquad (4.29)$$

Comment on the concept of the heat rate:

At the molecular level (say, microlevel), the exchange of energy between a system and its environment results from direct interaction of molecules, that is, from external forces acting on the system of molecules.

In a macroscopic description (say, macrolevel) we work with a continuous medium made of elementary particles of continuum. We do not look at internal structure of a particle. A size of the particle depends on a problem we want to solve. At the macrolevel we can describe (recognize) a portion of energy supplied by environment to volume V of continuum as a work of tractions acting on a surface of the volume. Obviously, what we recognize as stress and traction is determined by the size of the elementary particle. That portion of the energy exchange between volume V and its environment that cannot be described as (reduced to) a work of external tractions we must consider as a heat. This heat clearly is due to the internal structure of the considered elementary particle of continuum.

We can think of some macrolevel, say, intermediate macrolevel, between the molecular level and the very macrolevel we work with. At the intermediate macrolevel, rapid spatial and temporal variations of stresses corresponding to this intermediate macrolevel, do not affect magnitude of the stress at macrolevel. The work of stresses at the intermediate macrolevel has to be included as heat at the macrolevel. This is particularly true for the energy transferred by short waves recognizable at the intermediate macrolevel but smeared out at the macrolevel. Thus, term $\Delta q_i \nu_i$ in eqs. (4.27) and (4.29) includes the heat and radiation losses recognizable at the intermediate macrolevel.

An other aspect of the radiation loss is that the shorter the wavelength, the shorter a distance at which the short-wave radiation is transformed into heat due to propagation.

4.3 Energy Budget at a Fracture Edge

Consider now volume V intersecting fracture surface $\Sigma(t)$ over an area $\Delta\Sigma$ and containing fracture's edge, $\Delta L(t)$. The geometrical configuration of the problem is illustrated in Fig. 5. Due to the presence of the fracture's edge in volume V the integrands in the first integrals in eqs. (4.5), (4.6) and (4.16) might be singular at the fracture's

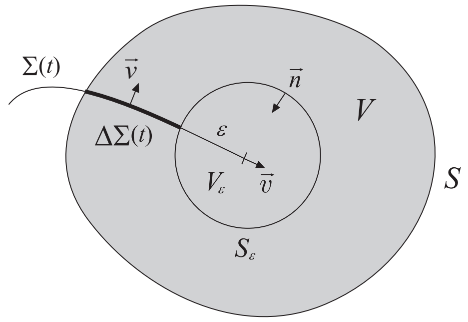

Figure 5. Material volume V intersecting fracture surface $\Sigma(t)$ and containing fracture's edge.

edge. Therefore, it is not possible to differentiate with respect to time under the integral sign. In order to avoid singularity, consider a small toroidal volume V_ε surrounding the fracture's edge. Then we can make use of relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \phi\,\mathrm{d}V \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\lim_{\varepsilon\to 0}\int_{V-V_\varepsilon}\phi\,\mathrm{d}V \;=\; \lim_{\varepsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}t}\int_{V-V_\varepsilon}\phi\,\mathrm{d}V\,. \tag{4.30}$$

However, because the fracture's edge propagates, the small toroidal volume V_ε propagates with the edge and the volume depends on time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \phi\,\mathrm{d}V \;=\; \lim_{\varepsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}t}\int_{V-V_\varepsilon(t)}\phi\,\mathrm{d}V\,. \tag{4.31}$$

Then

$$\lim_{\varepsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}t}\int_{V-V_\varepsilon(t)}\phi\,\mathrm{d}V \;=\; \lim_{\varepsilon\to 0}\int_{V-V_\varepsilon(t)}\dot\phi\,\mathrm{d}V \;+\; \lim_{\varepsilon\to 0}\int_{S+S_\varepsilon(t)}\phi\,\dot x_j\,n_j\,\mathrm{d}S\,. \tag{4.32}$$

Because $\dot x_j = 0$ on S and denoting the velocity of the fracture's edge $v_j = \dot x_j$, we obtain finally for our volume V

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \phi\,\mathrm{d}V \;=\; \lim_{\varepsilon\to 0}\int_{V-V_\varepsilon(t)}\dot\phi\,\mathrm{d}V \;+\; \lim_{\varepsilon\to 0}\int_{S_\varepsilon(t)}\phi\,v_j\,n_j\,\mathrm{d}S\,. \tag{4.33}$$

We apply relation (4.33) to time rates of the internal and kinetic energies:

$$\begin{aligned}
&\dot U_V + \dot K + \dot U_\Sigma \;=\\
&= \lim_{\varepsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{V-V_\varepsilon(t)}\left(\mathcal{U} + \frac{1}{2}\rho\,\dot u_i\,\dot u_i\right)\mathrm{d}V \;+\; \int_{\Delta\Sigma(t)} 2\gamma\,\mathrm{d}S\right] \;=\\
&= \lim_{\varepsilon\to 0}\left\{\int_{V-V_\varepsilon(t)}\left[\dot{\mathcal{U}} + \frac{1}{2}\rho\,\frac{\partial}{\partial t}\left(\dot u_i\,\dot u_i\right)\right]\mathrm{d}V \;+\; \int_{\Delta\Sigma(t)} 2\dot\gamma\,\mathrm{d}S \;+\right.\\
&\left. +\; \int_{S_\varepsilon(t)}\left(\mathcal{U} + \frac{1}{2}\rho\,\dot u_i\,\dot u_i\right) v_j\,n_j^\varepsilon\,\mathrm{d}S\right\} \;+\; \int_{\Delta L(t)} 2\gamma\,v\,\mathrm{d}L\,.
\end{aligned} \tag{4.34}$$

Here $v = (v_i\, v_i)^{\frac{1}{2}}$. For the rate of work of external forces we can write

$$\begin{aligned}
\dot{A} &= \lim_{\varepsilon \to 0} \left[\int_S \sigma_{ij}\, \dot{u}_i\, n_j\, \mathrm{d}S + \int_{V-V_\varepsilon(t)} f_i\, \dot{u}_i\, \mathrm{d}V \right] = \\
&= \lim_{\varepsilon \to 0} \left[\int_{V-V_\varepsilon(t)} (\, \sigma_{ij}\, \dot{u}_i\,)_{,j}\, \mathrm{d}V - \int_{S_\varepsilon} \sigma_{ij}\, \dot{u}_i\, n_j^\varepsilon\, \mathrm{d}S + \right. \\
&+ \left. \int_{\Delta\Sigma(t)} \sigma_{ij}\, \Delta\dot{u}_i\, \nu_j\, \mathrm{d}S + \int_{V-V_\varepsilon(t)} f_i\, \dot{u}_i\, \mathrm{d}V \right] ,
\end{aligned} \tag{4.35}$$

$$\begin{aligned}
\dot{A} &= \lim_{\varepsilon \to 0} \left\{ \int_{V-V_\varepsilon(t)} [\, (\, \sigma_{ij}\, \dot{u}_i\,)_{,j} + f_i\, \dot{u}_i\,]\, \mathrm{d}V - \right. \\
&- \left. \int_{S_\varepsilon} \sigma_{ij}\, \dot{u}_i\, n_j^\varepsilon\, \mathrm{d}S + \int_{\Delta\Sigma(t)} \sigma_{ij}\, \Delta\dot{u}_i\, \nu_j\, \mathrm{d}S \right\} ,
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
\dot{A} &= \lim_{\varepsilon \to 0} \left\{ \int_{V-V_\varepsilon(t)} \left[\frac{1}{2}\, \rho\, \frac{\partial}{\partial t}\, (\, \dot{u}_i\, \dot{u}_i\,) + \sigma_{ij}\, \dot{u}_{i,j} \right] \mathrm{d}V - \right. \\
&- \left. \int_{S_\varepsilon} \sigma_{ij}\, \dot{u}_i\, n_j^\varepsilon\, \mathrm{d}S + \int_{\Delta\Sigma(t)} \sigma_{ij}\, \Delta\dot{u}_i\, \nu_j\, \mathrm{d}S \right\} .
\end{aligned} \tag{4.37}$$

At the r.h.s. of the 1st line in eq. (4.35) we clearly recognize the work of external forces: the first integral is the rate of doing mechanical work by tractions at surface S, the second integral is the rate of doing mechanical work by body forces throughout volume $V - V_\varepsilon$. At the r.h.s. of eq. (4.37) we recognize the rates of change of the kinetic energy and energy of deformation in volume $V - V_\varepsilon$, and rates of doing mechanical work by tractions at surface S_ε and tractions at surface $\Delta\Sigma$ against friction.

In analogy to the rate of work of tractions at surface S, we have for the rate of heating

$$\begin{aligned}
\dot{Q} &= \lim_{\varepsilon \to 0} \left[- \int_S q_i\, n_i\, \mathrm{d}S \right] = \\
&= \lim_{\varepsilon \to 0} \left[- \int_{V-V_\varepsilon(t)} q_{i,i}\, \mathrm{d}V + \int_{S_\varepsilon} q_i\, n_i^\varepsilon\, \mathrm{d}S - \right. \\
&- \left. \int_{\Delta\Sigma(t)} \Delta q_i\, \nu_i\, \mathrm{d}S \right] .
\end{aligned} \tag{4.38}$$

Substitution of eqs. (4.34), (4.37) and (4.38) into eq. (4.18) yields

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \left\{ \int_{V-V_\varepsilon(t)} \left[\sigma_{ij}\, \dot{u}_{i,j} \; - q_{i,i} \; - \dot{\mathcal{U}} \right] \mathrm{d}V \; + \right. \\
&+ \quad \int_{S_\varepsilon} \left[-\, \sigma_{ij}\, \dot{u}_j \; + \; q_i \; - \; \left(\mathcal{U} \; + \; \frac{1}{2}\, \rho\, \dot{u}_j\, \dot{u}_j \right) v_i \right] n_i^\varepsilon \, \mathrm{d}S \; + \\
&+ \quad \left. \int_{\Delta\Sigma(t)} [\, \sigma_{ij}\, \nu_j\, \Delta\dot{u}_i \; - \; \Delta q_i\, \nu_i \; - \; 2\dot{\gamma} \,] \; \mathrm{d}S \right\} \; - \\
&- \quad \int_{\Delta L(t)} 2\gamma\, v \, \mathrm{d}L \;\; = \;\; 0\,.
\end{aligned}
\tag{4.39}
$$

The first and third terms in eq. (4.39) vanish due to eqs. (4.8) and (4.27), respectively. Then eq. (4.39) reduces to

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{S_\varepsilon} \left[-\, \sigma_{ij}\, \dot{u}_j \; + \; q_i \; - \; \left(\mathcal{U} \; + \; \frac{1}{2}\, \rho\, \dot{u}_j\, \dot{u}_j \right) v_i \right] n_i^\varepsilon \, \mathrm{d}S \; - \\
&- \quad \int_{\Delta L(t)} 2\gamma\, v \, \mathrm{d}L \;\; = \;\; 0\,.
\end{aligned}
\tag{4.40}
$$

Denote by l_ε the curve obtained as the cross-section of surface S_ε normal to the fracture's edge. Then

$$
\int_{S_\varepsilon} \phi \, \mathrm{d}S \;\; = \;\; \int_{\Delta L(t)} \left(\int_{l_\varepsilon} \phi \, \mathrm{d}l \right) \mathrm{d}L
\tag{4.41}
$$

and eq. (4.40) can be rewritten

$$
\begin{aligned}
&\int_{\Delta L(t)} \left\{ 2\gamma\, v \;\; + \;\; \lim_{\varepsilon\to 0} \int_{l_\varepsilon} \left[\sigma_{ij}\, \dot{u}_j \; - \; q_i \; + \right. \right. \\
&\qquad\qquad \left. \left. + \; \left(\mathcal{U} \; + \; \frac{1}{2}\, \rho\, \dot{u}_j\, \dot{u}_j \right) v_i \right] n_i^\varepsilon \, \mathrm{d}l \right\} \mathrm{d}L \;\; = \;\; 0\,.
\end{aligned}
\tag{4.42}
$$

Due to arbitrariness of volume V and, consequently, $\Delta L(t)$, and assuming continuity of the integrand,

$$
\begin{aligned}
&2\gamma\, v \;\; + \\
&+ \;\; \lim_{\varepsilon\to 0} \int_{l_\varepsilon} \left[\sigma_{ij}\, \dot{u}_j \; - \; q_i \; + \; \left(\mathcal{U} \; + \; \frac{1}{2}\, \rho\, \dot{u}_j\, \dot{u}_j \right) v_i \right] n_i^\varepsilon \, \mathrm{d}l \;\; = \;\; 0\,.
\end{aligned}
\tag{4.43}
$$

Integrate eq. (4.10):

$$
\mathcal{U} \;\; = \;\; \mathcal{U}_0 \; + \; \int_0^t (\, \sigma_{ij}\, \dot{\varepsilon}_{ij} \; - \; q_{i,i} \,) \; \mathrm{d}t\,.
\tag{4.44}
$$

Here, $\mathcal{U}_0$ is the volume density of internal energy at time $t = 0$. Substituting eq. (4.44) into eq. (4.43) we obtain

$$2\gamma\, v \;+\; \lim_{\varepsilon\to 0} \int_{l_\varepsilon} \Bigg\{ \sigma_{ij}\, \dot u_j \;-\; q_i \;+ \tag{4.45}$$

$$+ \;\left[\mathcal{U}_0 \;+\; \int_0^t (\,\sigma_{kj}\, \dot\varepsilon_{kj} \;-\; q_{j,j}\,)\ \mathrm{d}t \;+\; \frac{1}{2}\, \rho\, \dot u_j\, \dot u_j \right] v_i \Bigg\}\, n_i^\varepsilon\, \mathrm{d}l \;=\; 0\,.$$

Because $\mathcal{U}_0$ is finite at the fracture edge, in the limit for $\varepsilon \to 0$ the corresponding term in eq. (4.45) will vanish. Then

$$2\gamma\, v \;+\; \lim_{\varepsilon\to 0} \int_{l_\varepsilon} \left\{ \sigma_{ij}\, \dot u_j \;+\; v_i \left[\int_0^t \sigma_{kj}\, \dot\varepsilon_{kj}\, \mathrm{d}t \;+\; \frac{1}{2}\, \rho\, \dot u_j\, \dot u_j \right] \right\} n_i^\varepsilon\, \mathrm{d}l \;-$$

$$-\; \lim_{\varepsilon\to 0} \int_{l_\varepsilon} \left\{ q_i \;+\; v_i \int_0^t q_{j,j}\ \mathrm{d}t \right\} n_i^\varepsilon\, \mathrm{d}l \;=\; 0\,. \tag{4.46}$$

The second term in the equation, taken with the opposite sign, represents mechanical energy flux, that is work done at the fracture's edge.

5 Seismic Energy

Seismic energy is defined as the total energy transmitted by seismic waves through surface S_0 completely enclosing the source:

$$E_q \;=\; -\int_{t_0}^{t_1} \mathrm{d}t \int_{S_0} \tau_{ij}\, \dot u_i\, n_j\, \mathrm{d}S\,, \tag{5.1}$$

where t_1 is the source duration and the waves reflected from the free surface of the Earth are neglected. Definition of E_q by eq. (5.1) enables measurement of seismic energy. In practical applications S_0 is chosen as a sufficiently large sphere and seismogram is approximated by a set of sinusoids. (This yields a well-known Galitzin's formula in seismology.)

Let E_p be the potential energy in volume V and ΔE_p its change during the earthquake. The change in the potential energy consists of the changes in the elastic and gravitational energies,

$$\Delta E_p \;=\; \int_V \left(\, \mathcal{W}^1 \;-\; \mathcal{W}^0 \;-\; f_i\, u_i^1\,\right)\ \mathrm{d}V\,, \tag{5.2}$$

and, considering relation (2.25) for the density of the elastic energy,

$$\Delta E_p \;=\; \int_V \left[\frac{1}{2} \, (\sigma^1_{ij} \; + \; \sigma^0_{ij}) \, u^1_{i},_j \; - f_i \, u^1_i \right] \mathrm{d}V \; , \tag{5.3}$$

where f_i is the gravitational force. We consider that f_i during the earthquake does not change. We also consider states just before and after the earthquake as states of equilibrium.

Therefore,

$$0 \;=\; \sigma^0_{ij},_j \; + f_i \tag{5.4}$$

and

$$0 \;=\; \sigma^1_{ij},_j \; + f_i \; . \tag{5.5}$$

It follows from eqs. (5.4) and (5.5) that

$$\frac{1}{2} \, (\sigma^1_{ij},_j \; + \sigma^0_{ij},_j) \;=\; - \, f_i \; . \tag{5.6}$$

Using relation (5.6) the integrand in eq. (5.3) can be written as

$$\begin{aligned} & \frac{1}{2} \, (\sigma^1_{ij} \; + \; \sigma^0_{ij}) \, u^1_{i},_j \; + \frac{1}{2} \, (\sigma^1_{ij},_j \; + \sigma^0_{ij},_j) \, u^1_i \;= \\ = \; & \frac{1}{2} \, [\, (\sigma^1_{ij} \; + \; \sigma^0_{ij}) \, u^1_i \,],_j \; . \end{aligned} \tag{5.7}$$

Then we have

$$\Delta E_p \;=\; \frac{1}{2} \int_V \, [\, (\sigma^1_{ij} \; + \; \sigma^0_{ij}) \, u^1_i \,],_j \; \mathrm{d}V \; . \tag{5.8}$$

Assuming Σ_1 as the final ruptured area, the application of the Gauss's theorem yields

$$\Delta E_p \;=\; \frac{1}{2} \int_{S_0} (\sigma^1_{ij} + \sigma^0_{ij}) \, u^1_i \, n_j \, \mathrm{d}S \; - \; \frac{1}{2} \int_{\Sigma_1} (\sigma^1_{ij} + \sigma^0_{ij}) \, \Delta u^1_i \, n_j \, \mathrm{d}S \; . \tag{5.9}$$

The first integral in eq. (5.9) can be neglected if the radius of S_0 is sufficiently large. Then

$$\Delta E_p \;=\; - \, \frac{1}{2} \int_{\Sigma_1} (\sigma^1_{ij} + \sigma^0_{ij}) \, \Delta u^1_i \, n_j \, \mathrm{d}S \; . \tag{5.10}$$

Usually it is assumed that the decrease in the potential energy E_p is equal to the sum of the seismic energy E_q and the work of frictional forces A_f on the fault:

$$- \Delta E_p \;=\; E_q \; + \; A_f \; , \tag{5.11}$$

where

$$A_f \;=\; \int_{\Sigma_1} \sigma_{(f)i}\; \Delta u_i \;\mathrm{d}S \tag{5.12}$$

and $\sigma_{(f)i}$ is the magnitude of the frictional force. However, eq. (5.11) is essentially a new definition of the seismic energy. Equation (5.11) tries to define the seismic energy using the energy conservation law. The problem is, that such energy conservation law should be (even if it were assumed that the frictional work is the only loss of energy during the earthquake)

$$-\Delta E_p \;=\; \Delta E_{S_0} \;+\; A_f \;, \tag{5.13}$$

where ΔE_{S_0} is the energy released from volume V through its surface S_0 during the earthquake:

$$\Delta E_{S_0} \;=\; -\int_{t_0}^{t_1} \mathrm{d}t \int_{S_0} \sigma_{ij}\; n_j\; \dot{u}_i \;\mathrm{d}S \;. \tag{5.14}$$

Compared to the integral in (5.1), here the total stress σ_{ij} is in the integrand instead of the stress perturbation τ_{ij}. Consequently, the difference $\Delta E_{S_0} - E_q$ has to represent the work of the initial stress:

$$\Delta E_{S_0} \;-\; E_q \;=\; \int_{t_0}^{t_1} \mathrm{d}t \int_{S_0} \sigma^0_{ij}\; n_j\; \dot{u}_i \;\mathrm{d}S \;=\; \int_{S_0} \sigma^0_{ij}\; n_j\; u^1_i \;\mathrm{d}S \;, \tag{5.15}$$

since σ^0_{ij} does not depend on time. In general, the difference is non-zero. Kostrov and Das (1988) conclude that the non-zero difference means that definition (5.11) does not agree with definition (5.1).

In an effort to obtain E_q in relation to quantities characterizing the source, consider the rate of change of the elastic energy in volume V during the earthquake. In derivation of relation (2.25) we did not assume that state at time t_1 is the state of equilibrium. Therefore, instead of final time t_1 after the earthquake we can write eq. (2.25) for any time $t > t_0$ during the earthquake:

$$\mathcal{W}\,(\,t\,) \;=\; \mathcal{W}^0 \;+\; \frac{1}{2}\;\left(\,\sigma_{ij} \;+\; \sigma^0_{ij}\,\right)\; u_{i,j} \;\;. \tag{5.16}$$

The time rate of change of the elastic energy in volume V is then

$$\dot{W} \;=\; \frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{1}{2}\;\left(\,\sigma_{ij} \;+\; \sigma^0_{ij}\,\right)\; u_{i,j}\;\; \mathrm{d}V \;. \tag{5.17}$$

Volume V contains the propagating fracture's edge where stress and particle velocity have a singularity (of order $\frac{1}{2}$). Therefore, the time differentiation cannot be directly applied to the integrand. As in Chapter 4, the integral in eq. (5.17) can be written as

$$\lim_{\varepsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}t} \int_{V - V_\varepsilon(t)} \frac{1}{2} \left(\sigma_{ij} + \sigma^0_{ij} \right) u_{i,j} \ \mathrm{d}V \, . \tag{5.18}$$

Then

$$\begin{aligned} \dot{W} &= \lim_{\varepsilon \to 0} \int_{V - V_\varepsilon} \left[\frac{1}{2} \dot{\sigma}_{ij} \, u_{i,j} + \frac{1}{2} \left(\sigma_{ij} + \sigma^0_{ij} \right) \dot{u}_{i,j} \right] \mathrm{d}V + \\ &+ \lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{1}{2} \sigma_{ik} \, u_{i,k} \, v_j \, n_j \, \mathrm{d}S \, . \end{aligned} \tag{5.19}$$

Note that, Kostrov and Das (1988) in their equation on p. 153 have a negative sign in front of the surface integral. That and following negative signs on p. 154 can be explained by a change in the orientation of the normal to surface S_ε. We think that Kostrov and Das (1988), inconsistently with their Chapter 2, consider in Chapter 4 normal oriented outward volume V_ε. In our derivation we use normal as shown in Fig. 5.

Using $\sigma_{ij} = \sigma^0_{ij} + \tau_{ij}$, $\tau_{ij} = c_{ijkl} \, u_{k,l}$ and symmetry $c_{ijkl} = c_{klij}$ the integrand in the volume integral can be rewritten. Then

$$\begin{aligned} \dot{W} &= \lim_{\varepsilon \to 0} \int_{V - V_\varepsilon} \left(\tau_{ij} + \sigma^0_{ij} \right) \dot{u}_{i,j} \ \mathrm{d}V + \\ &+ \lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{1}{2} \sigma_{ik} \, u_{i,k} \, v_j \, n_j \, \mathrm{d}S \, . \end{aligned} \tag{5.20}$$

Recalling equation of motion for τ_{ij}, eq. (2.29), and equation of equilibrium for σ^0_{ij}, eq. (2.26), the integrand in the volume integral can be further modified

$$\begin{aligned} \left(\tau_{ij} + \sigma^0_{ij} \right) \dot{u}_{i,j} &= \left[\left(\tau_{ij} + \sigma^0_{ij} \right) \dot{u}_i \right]_{,j} - \left(\tau_{ij} + \sigma^0_{ij} \right)_{,j} \dot{u}_i = \\ &= \left(\sigma_{ij} \, \dot{u}_i \right)_{,j} - \left(\tau_{ij,j} + \sigma^0_{ij,j} \right) \dot{u}_i = \\ &= \left(\sigma_{ij} \, \dot{u}_i \right)_{,j} - \rho \, \ddot{u}_i \, \dot{u}_i + f_i \, \dot{u}_i \, . \end{aligned} \tag{5.21}$$

Then

$$\begin{aligned} \dot{W} &= \lim_{\varepsilon \to 0} \int_{V - V_\varepsilon} \left[\left(\sigma_{ij} \, \dot{u}_i \right)_{,j} - \rho \, \dot{u}_i \, \ddot{u}_i + f_i \, \dot{u}_i \right] \mathrm{d}V \\ &+ \lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{1}{2} \sigma_{ik} \, u_{i,k} \, v_j \, n_j \, \mathrm{d}S \, . \end{aligned} \tag{5.22}$$

The volume integral can be split into three integrals. The first integral can be rewritten using Gauss's divergence theorem:

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{V-V_\varepsilon} (\, \sigma_{ij}\, \dot{u}_i \,)_{,j} \; \mathrm{d}V \;\; &= \int_{S_0} \sigma_{ij}\, \dot{u}_i\, n_j\, \mathrm{d}S \; - \\ &- \int_{\Sigma(t)} \sigma_{ij}\, \Delta\dot{u}_i\, n_j\, \mathrm{d}S \; + \; \lim_{\varepsilon\to 0} \int_{S_\varepsilon} \sigma_{ij}\, \dot{u}_i\, n_j\, \mathrm{d}S \; . \end{aligned} \tag{5.23}$$

Consider the rate of change of the kinetic energy K:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} K \;\; &= \;\; \frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{1}{2}\, \rho\, \dot{u}_i\, \dot{u}_i\, \mathrm{d}V \;\; = \\ &= \;\; \frac{\mathrm{d}}{\mathrm{d}t} \lim_{\varepsilon\to 0} \int_{V-V_\varepsilon} \frac{1}{2}\, \rho\, \dot{u}_i\, \dot{u}_i\, \mathrm{d}V \;\; = \\ &= \;\; \lim_{\varepsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}t} \int_{V-V_\varepsilon} \frac{1}{2}\, \rho\, \dot{u}_i\, \dot{u}_i\, \mathrm{d}V \;\; = \\ &= \;\; \lim_{\varepsilon\to 0} \int_{V-V_\varepsilon} \rho\, \dot{u}_i\, \ddot{u}_i\, \mathrm{d}V \; + \; \lim_{\varepsilon\to 0} \int_{S_\varepsilon} \frac{1}{2}\, \rho\, \dot{u}_i\, \dot{u}_i\, v_j\, n_j\, \mathrm{d}S \; . \end{aligned} \tag{5.24}$$

The second integral then can be written as

$$- \lim_{\varepsilon\to 0} \int_{V-V_\varepsilon} \rho\, \dot{u}_i\, \ddot{u}_i\, \mathrm{d}V \;\; = \;\; - \left(\frac{\mathrm{d}}{\mathrm{d}t} K \; - \; \lim_{\varepsilon\to 0} \int_{S_\varepsilon} \frac{1}{2}\, \rho\, \dot{u}_i\, \dot{u}_i\, v_j\, n_j\, \mathrm{d}S \right). \tag{5.25}$$

Consider the time rate of change of the gravitational energy of volume V:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_g \;\; = \;\; \frac{\mathrm{d}}{\mathrm{d}t} \int_V (\, -f_i\, u_i) \;\; \mathrm{d}V \;\; = \;\; \lim_{\varepsilon\to 0} \int_{V-V_\varepsilon} (\, -f_i\, \dot{u}_i) \;\; \mathrm{d}V \; . \tag{5.26}$$

Equation (5.22) can be now written in the form

$$\begin{aligned} \dot{W} \;\; = \;\; & \int_{S_0} \sigma_{ij}\, \dot{u}_i\, n_j\, \mathrm{d}S \; - \int_{\Sigma(t)} \sigma_{ij}\, \Delta\dot{u}_i\, n_j\, \mathrm{d}S \; + \; \lim_{\varepsilon\to 0} \int_{S_\varepsilon} \sigma_{ij}\, \dot{u}_i\, n_j\, \mathrm{d}S \\ - \;\; & \frac{\mathrm{d}}{\mathrm{d}t} K \; + \; \lim_{\varepsilon\to 0} \int_{S_\varepsilon} \frac{1}{2}\, \rho\, \dot{u}_i\, \dot{u}_i\, v_j\, n_j\, \mathrm{d}S \; - \; \frac{\mathrm{d}}{\mathrm{d}t} E_g \\ + \;\; & \lim_{\varepsilon\to 0} \int_{S_\varepsilon} \frac{1}{2}\, \sigma_{ik}\, u_{i,k}\, v_j\, n_j\, \mathrm{d}S \; , \end{aligned} \tag{5.27}$$

$$\begin{aligned} \dot{W} \;\; = \;\; & - \dot{K} \; - \; \dot{E}_g \; + \int_{S_0} \sigma_{ij}\, \dot{u}_i\, n_j\, \mathrm{d}S \; - \int_{\Sigma(t)} \sigma_{ij}\, \Delta\dot{u}_i\, n_j\, \mathrm{d}S \; + \\ + \;\; & \lim_{\varepsilon\to 0} \int_{S_\varepsilon} \left(\sigma_{ij}\, \dot{u}_i\, n_j \; + \; \frac{1}{2}\, \rho\, \dot{u}_i\, \dot{u}_i\, v_j\, n_j \; + \; \frac{1}{2}\, \sigma_{ik}\, u_{i,k}\, v_j\, n_j \right) \; \mathrm{d}S \; . \end{aligned} \tag{5.28}$$

Kostrov and Das (1988) denote the last term as

$$- \dot{E}_{\gamma_{eff}} \quad = \quad - 2 \, \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Sigma(t)} \gamma_{eff} \; \mathrm{d}S \; . \tag{5.29}$$

Integration of eq. (5.28) from time t_0 until time t_1 gives

$$\begin{aligned} \Delta \; (W \; + \; E_g \;) \; + \; \Delta \, E_{\gamma_{eff}} \quad = & \quad \int_{t_0}^{t_1} \mathrm{d}t \int_{S_0} \sigma_{ij} \; \dot{u}_i \; n_j \; \mathrm{d}S \; - \\ & - \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} \sigma_{ij} \; \Delta \dot{u}_i \; n_j \; \mathrm{d}S \; . \end{aligned} \tag{5.30}$$

$\Delta \, E_{\gamma_{eff}}$ is the fracture work and the kinetic energies at time t_0 and t_1 are zero because both states are states of equilibrium. The first term is the change of the potential energy, $\Delta \, E_p \; = \; \Delta \, (\, W \; + \; E_g \,)$. The first term on the r.h.s. is, according to eq. (5.14), $-\Delta \, E_{S_0}$. The second term on the r.h.s. is $-A_f$. Thus, eq. (5.30) can be written as

$$\Delta \, E_p \; + \; \Delta \, E_{\gamma_{eff}} \; + \; \Delta \, E_{S_0} \; + \; A_f \; = \; 0 \; . \tag{5.31}$$

Compared to eq. (5.13), eq. (5.31) includes not only frictional losses but also the energy dissipation due to fracture.

The first term on the r.h.s. of eq. (5.30) can be rewritten as

$$\begin{aligned} \int_{t_0}^{t_1} \mathrm{d}t \int_{S_0} \sigma_{ij} \; \dot{u}_i \; n_j \; \mathrm{d}S \quad = & \quad \int_{t_0}^{t_1} \mathrm{d}t \int_{S_0} \left(\sigma_{ij} \; - \; \sigma_{ij}^0 \right) \; \dot{u}_i \; n_j \; \mathrm{d}S \; + \\ & + \int_{t_0}^{t_1} \mathrm{d}t \int_{S_0} \sigma_{ij}^0 \; \dot{u}_i \; n_j \; \mathrm{d}S \quad = \\ & = \int_{t_0}^{t_1} \mathrm{d}t \int_{S_0} \left(\sigma_{ij} \; - \; \sigma_{ij}^0 \right) \; \dot{u}_i \; n_j \; \mathrm{d}S \; + \\ & + \int_{S_0} \sigma_{ij}^0 \; u_i^1 \; n_j \; \mathrm{d}S \; , \end{aligned} \tag{5.32}$$

since $u_i \; (t_0) \; = \; 0$. Considering eq. (5.1), eq. (5.32) gives

$$\int_{t_0}^{t_1} \mathrm{d}t \int_{S_0} \sigma_{ij} \; \dot{u}_i \; n_j \; \mathrm{d}S \quad = \quad -E_q \; + \int_{S_0} \sigma_{ij}^0 \; u_i^1 \; n_j \; \mathrm{d}S \; . \tag{5.33}$$

Substitution of eq. (5.33) into eq. (5.30) gives

$$\begin{aligned} \Delta \, E_p \; + \; \Delta \, E_{\gamma_{eff}} \quad = & \quad - \; E_q \; + \int_{S_0} \sigma_{ij}^0 \; u_i^1 \; n_j \; \mathrm{d}S \; - \\ & - \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} \sigma_{ij} \; \Delta \dot{u}_i \; n_j \; \mathrm{d}S \; . \end{aligned} \tag{5.34}$$

Substitution of relation for the change of the potential energy, eq. (5.9), into eq. (5.34) gives

$$
\begin{aligned}
E_q \;&=\; -\,\Delta\, E_{\gamma_{eff}} \,+\, \int_{S_0} \sigma^0_{ij}\, u^1_i\, n_j\, \mathrm{d}S \;-\; \frac{1}{2}\int_{S_0} (\, \sigma^1_{ij} \,+\, \sigma^0_{ij}\,)\, u^1_i\, n_j\, \mathrm{d}S \\
&\quad -\, \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} \sigma_{ij}\, \Delta\dot{u}_i\, n_j\, \mathrm{d}S \;+\; \frac{1}{2}\int_{\Sigma_1} (\, \sigma^1_{ij} \,+\, \sigma^0_{ij}\,)\, \Delta u^1_i\, n_j\, \mathrm{d}S\;,
\end{aligned}
\tag{5.35}
$$

$$
\begin{aligned}
E_q \;&=\; \frac{1}{2}\int_{S_0} (\, \sigma^0_{ij} \,-\, \sigma^1_{ij}\,)\, u^1_i\, n_j\, \mathrm{d}S \\
&\quad -\, \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} \sigma_{ij}\, \Delta\dot{u}_i\, n_j\, \mathrm{d}S \;+\; \frac{1}{2}\int_{\Sigma_1} (\, \sigma^1_{ij} \,+\, \sigma^0_{ij}\,)\, \Delta u^1_i\, n_j\, \mathrm{d}S \\
&\quad -\, \Delta\, E_{\gamma_{eff}}\;.
\end{aligned}
\tag{5.36}
$$

Equation (5.36) corresponds to eq. (4.4.21) of Kostrov and Das (1988) except the sign in the third term on the r.h.s. Equation (4.4.21) has wrong '$-$' sign in the integrand.

The second and third integrals in eq. (5.36) can be rewritten:

$$
\begin{aligned}
&-\int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} \sigma_{ij}\, \Delta\dot{u}_i\, n_j\, \mathrm{d}S \;+\; \frac{1}{2}\int_{\Sigma_1} (\, \sigma^1_{ij} \,+\, \sigma^0_{ij}\,)\, \Delta u^1_i\, n_j\, \mathrm{d}S \;= \\
&=\; -\int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} \sigma^0_{ij}\, \Delta\dot{u}_i\, n_j\, \mathrm{d}S \;-\; \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} \tau_{ij}\, \Delta\dot{u}_i\, n_j\, \mathrm{d}S \;+ \\
&\qquad\qquad +\; \frac{1}{2}\int_{\Sigma_1} (\, \sigma^1_{ij} \,+\, \sigma^0_{ij}\,)\, \Delta u^1_i\, n_j\, \mathrm{d}S\;.
\end{aligned}
\tag{5.37}
$$

The first integral on the r.h.s can be rewritten as

$$
-\int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(\,t\,)} \sigma^0_{ij}\, \Delta\dot{u}_i\, n_j\, \mathrm{d}S \;=\; -\int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma_1} \dot{\Phi}_j\,(\,t\,)\, n_j\, \mathrm{d}S\;,
\tag{5.38}
$$

where

$$
\begin{aligned}
\dot{\Phi}_j\,(\,t\,) \;&=\; \sigma^0_{ij}\, \Delta\dot{u}_i\,(\,t\,) \qquad && at\;\; \Sigma\,(\,t\,) \\
&=\; 0 && at\;\; \Sigma_1 \,-\, \Sigma\,(\,t\,)\;\;,
\end{aligned}
\tag{5.39}
$$

that is,

$$
\begin{aligned}
\Phi_j\,(\,t_1\,) \;&=\; \sigma^0_{ij}\, \Delta u^1_i \qquad && at\;\; \Sigma_1\,, \\
\Phi_j\,(\,t_0\,) \;&=\; 0 && at\;\; \Sigma_1\,,
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
-\int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma_1} \dot{\Phi}_j\,(\,t\,)\, n_j\, \mathrm{d}S \;&=\; -\int_{\Sigma_1} \mathrm{d}S \int_{t_0}^{t_1} \dot{\Phi}_j\,(\,t\,)\;\, n_j\, \mathrm{d}t \\
&=\; -\int_{\Sigma_1} \mathrm{d}S\;\, [\,\Phi_j\,(\,t_1\,) \;-\; \Phi_j\,(\,t_0\,)\,]\;\, n_j \\
&=\; -\int_{\Sigma_1} \sigma^0_{ij}\, \Delta u^1_i\, n_j\, \mathrm{d}S\;.
\end{aligned}
\tag{5.41}
$$

Thus we have from eqs. (5.38), (5.40) and (5.41) that

$$- \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(\,t\,)} \sigma_{ij}^0 \, \Delta \dot{u}_i \, n_j \, \mathrm{d}S \quad = \quad - \int_{\Sigma_1} \sigma_{ij}^0 \, \Delta u_i^1 \, n_j \, \mathrm{d}S \, . \tag{5.42}$$

Substitution of eq. (5.42) into the r.h.s. of eq. (5.37) gives

$$\begin{aligned} & - \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} \sigma_{ij} \, \Delta \dot{u}_i \, n_j \, \mathrm{d}S \; + \; \frac{1}{2} \int_{\Sigma_1} (\, \sigma_{ij}^1 \; + \; \sigma_{ij}^0 \,) \, \Delta u_i^1 \, n_j \, \mathrm{d}S \; = \\ & = \; - \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} \tau_{ij} \, \Delta \dot{u}_i \, n_j \, \mathrm{d}S \; + \; \frac{1}{2} \int_{\Sigma_1} (\, \sigma_{ij}^1 \; - \; \sigma_{ij}^0 \,) \, \Delta u_i^1 \, n_j \, \mathrm{d}S \, . \end{aligned} \tag{5.43}$$

Substitution of eq. (5.43) into eq. (5.36) gives

$$\begin{aligned} E_q \; = \; & \frac{1}{2} \int_{S_0} (\, \sigma_{ij}^0 \; - \; \sigma_{ij}^1 \,) \, u_i^1 \, n_j \, \mathrm{d}S \; - \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} (\, \sigma_{ij} \; - \; \sigma_{ij}^0 \,) \; \Delta \dot{u}_i \, n_j \, \mathrm{d}S \\ & + \; \frac{1}{2} \int_{\Sigma_1} (\, \sigma_{ij}^1 \; - \; \sigma_{ij}^0 \,) \, \Delta u_i^1 \, n_j \, \mathrm{d}S \; - \; \Delta \, E_{\gamma_{eff}} \, , \end{aligned} \tag{5.44}$$

where we replaced τ_{ij} by $\sigma_{ij} \; - \; \sigma_{ij}^0$ in the second integral on the r.h.s. Equation (5.44) means that the seismic energy E_q does not depend separately on σ_{ij} or σ_{ij}^0 but only on the stress perturbation $\tau_{ij} \; = \; \sigma_{ij} \; - \; \sigma_{ij}^0$.

An other aspect of relation (5.44) is that the seismic energy depends on the choice of surface S_0 which is not a characteristic parameter of the earthquake source itself.

Because the static displacements decrease with distance R from the source faster than d/R, where d is the source dimension, and the final stress perturbation is expressed by the first derivative of u_i^1 and thus decreases faster than d/R^2, choosing for S_0 a sphere of a sufficiently large radius R it is found that the first term in eq. (5.44) decreases faster than d/R and consequently may be made arbitrarily small. Thus, if S_0 is chosen in this way, relation (5.44) becomes

$$\begin{aligned} E_q \; = \; & \frac{1}{2} \int_{\Sigma_1} (\, \sigma_{ij}^1 \; - \; \sigma_{ij}^0 \,) \, \Delta u_i^1 \, n_j \, \mathrm{d}S \; - \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} (\, \sigma_{ij} \; - \; \sigma_{ij}^0 \,) \, \Delta \dot{u}_i \, n_j \, \mathrm{d}S \\ & - \; \Delta \, E_{\gamma_{eff}} \, , \end{aligned} \tag{5.45}$$

or, putting

$$\Delta \, E_{\gamma_{eff}} \quad = \quad 2 \, \gamma_{eff} \, S \, , \tag{5.46}$$

where S is the area of Σ_1,

$$E_q \;=\; \frac{1}{2}\int_{\Sigma_1}(\,\sigma^1_{ij} \,-\, \sigma^0_{ij}\,)\;\Delta u^1_i\; n_j\;\mathrm{d}S \;-\; \int_{t_0}^{t_1}\mathrm{d}t\int_{\Sigma(t)}(\,\sigma_{ij} \,-\, \sigma^0_{ij}\,)\;\Delta\dot{u}_i\; n_j\;\mathrm{d}S \tag{5.47}$$
$$-\;\; 2\,\gamma_{eff}\; S\;.$$

(Equation (5.47) corresponds to eq. (4.4.23) in Kostrov and Das (1988)).

The integrand in the second integral on the r.h.s. of eq. (5.47) can be written as

$$\begin{aligned}(\,\sigma_{ij} \,-\, \sigma^0_{ij}\,)\;\Delta\dot{u}_i\; n_j \;&=\; \frac{\mathrm{d}}{\mathrm{d}t}\,[\;(\,\sigma_{ij} \,-\, \sigma^0_{ij}\,)\;\Delta u_i\;]\; n_j \;-\\ &-\;\left[\,\frac{\mathrm{d}}{\mathrm{d}t}\,(\,\sigma_{ij} \,-\, \sigma^0_{ij}\,)\,\right]\;\Delta u_i\; n_j \;\;=\\ &=\;\frac{\mathrm{d}}{\mathrm{d}t}\,[\;(\,\sigma_{ij} \,-\, \sigma^0_{ij}\,)\;\Delta u_i\;]\; n_j \;\;-\;\; \dot{\sigma}_{ij}\;\Delta u_i\; n_j\;.\end{aligned} \tag{5.48}$$

The second integral is then

$$\begin{aligned}&-\int_{t_0}^{t_1}\mathrm{d}t\int_{\Sigma(t)}(\,\sigma_{ij} \,-\, \sigma^0_{ij}\,)\;\Delta\dot{u}_i\; n_j\;\mathrm{d}S \;\;=\\ =\;\;&-\;\int_{t_0}^{t_1}\mathrm{d}t\int_{\Sigma(t)}\frac{\mathrm{d}}{\mathrm{d}t}\,[\;(\,\sigma_{ij} \,-\, \sigma^0_{ij}\,)\;\Delta u_i\;]\; n_j\;\mathrm{d}S \;+\\ &+\;\int_{t_0}^{t_1}\mathrm{d}t\int_{\Sigma(t)}\dot{\sigma}_{ij}\;\Delta u_i\; n_j\;\mathrm{d}S\;.\end{aligned} \tag{5.49}$$

Evaluate the time integral in the first term on the r.h.s..

$$\begin{aligned}&-\int_{t_0}^{t_1}\mathrm{d}t\int_{\Sigma(t)}\frac{\mathrm{d}}{\mathrm{d}t}\,[\;(\,\sigma_{ij} \,-\, \sigma^0_{ij}\,)\;\Delta u_i\;]\; n_j\;\mathrm{d}S \;\;=\\ =\;\;&-\int_{t_0}^{t_1}\mathrm{d}t\int_{\Sigma_1}\dot{\Phi}_j\;(\,t\,)\; n_j\;\mathrm{d}S\;,\end{aligned} \tag{5.50}$$

where

$$\begin{aligned}\dot{\Phi}_j\;(\,t\,)\;\;&=\;\;\frac{\mathrm{d}}{\mathrm{d}t}\,[\;(\,\sigma_{ij} \;\;-\;\; \sigma^0_{ij}\,)\;\Delta u_i\;] \qquad at\;\;\; \Sigma\;(\,t\,)\\ &=\;\;0 \qquad\qquad\qquad\qquad\qquad\quad at\;\;\; \Sigma_1 \,-\, \Sigma\;(\,t\,)\;\;,\end{aligned} \tag{5.51}$$

that is,

$$\begin{aligned}\Phi_j\;(\,t_1\,)\;\;&=\;\;(\,\sigma^1_{ij} \;\;-\;\; \sigma^0_{ij}\,)\;\Delta u^1_i \qquad at\;\;\; \Sigma_1\;,\\ \Phi_j\;(\,t_0\,)\;\;&=\;\;0 \qquad\qquad\qquad\qquad\quad at\;\;\; \Sigma_1\;.\end{aligned} \tag{5.52}$$

$$
\begin{aligned}
- \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma_1} \dot{\Phi}_j\,(\,t\,)\, n_j \, \mathrm{d}S \;&=\; - \int_{\Sigma_1} \mathrm{d}S \int_{t_0}^{t_1} \dot{\Phi}_j\,(\,t\,)\;\, n_j \, \mathrm{d}t \;\;= \\
&= - \int_{\Sigma_1} \mathrm{d}S \;\; [\, \Phi_j\,(\,t_1\,) - \Phi_j\,(\,t_0\,)\,] \;\; n_j \;\;= \\
&= - \int_{\Sigma_1} (\sigma_{ij}^1 \;-\; \sigma_{ij}^0\,)\; \Delta u_i^1 \, n_j \, \mathrm{d}S \;.
\end{aligned}
\tag{5.53}
$$

Then, using eqs. (5.49), (5.50) and (5.53) we can write relation (5.47) as

$$
\begin{aligned}
E_q \;\;=\;\; -\;\; & \frac{1}{2} \int_{\Sigma_1} (\,\sigma_{ij}^1 - \sigma_{ij}^0\,)\; \Delta u_i^1 \, n_j \, \mathrm{d}S \;\;+\; \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} \dot{\sigma}_{ij} \; \Delta u_i \, n_j \, \mathrm{d}S \;- \\
-\;\; & 2\, \gamma_{eff}\; S
\end{aligned}
\tag{5.54}
$$

or, as in Kostrov and Das (1988), eq. (4.4.24),

$$
\begin{aligned}
E_q \;\;=\;\; +\;\; & \frac{1}{2} \int_{\Sigma_1} (\,\sigma_{ij}^0 - \sigma_{ij}^1\,)\; \Delta u_i^1 \, n_j \, \mathrm{d}S \;\;+\; \int_{t_0}^{t_1} \mathrm{d}t \int_{\Sigma(t)} \dot{\sigma}_{ij} \; \Delta u_i \, n_j \, \mathrm{d}S \;- \\
-\;\; & 2\, \gamma_{eff}\; S \;.
\end{aligned}
\tag{5.55}
$$

Equation (5.55) is the final relation for seismic energy E_q in terms of the source parameters. The second term in the equation is called the Kostrov term (Cocco et al., 2007).

6 Realistic Fault-zone Model and Dynamic Weakening Processes

In our future work (diploma thesis) we will focus on generalization of the thermodynamics of earthquake rupture to the finite-thickness fault zone model. Such a generalization has to account for important well established facts of the earthquake source dynamics. As we mentioned in section on the breakdown zone model, one of the key aspects of the earthquake source dynamics is the understanding of the well established fact of the stress degradation with an increasing slip, that is process of the dynamic weakening. The dynamic weakening processes are becoming a subject of intensive research related to the realistic finite-thickness fault zone model (illustrated in Fig. 2). Rice and Cocco

(2007) present probably the first comprehensive review of results of field, laboratory and theoretical investigations of the dynamic weakening processes. Here we provide a concise outline of the basic mechanisms presented in a greater detail in paper by Rice and Cocco (2007).

These mechanisms have to be investigated as possible candidates for the dynamic weakening given the fact the earthquake slips are often accommodated within relatively very thin (mm) zones. Assuming an adiabatic shearing, large rapid slip (equal or larger than 1 m and more than 0.1 m/s) would lead to the increase in temperature larger than approximately 1000°C. Such a sudden increase in temperature would lead to melting and, assuming relatively low viscosity, to a mechanical lubrication of the fault. The lubrication would decrease the coefficient of friction and lead to decrease of shear traction, that is to the dynamic weakening. At the same time, melting would lead to formation of pseudotachylites. The problem is that pseudotachylites or other indications of the frictional melting are only very rarely found in the fault zones. One possible explanation might be that the pseudotachylites are not preserved in the mature faults. An other and more likely explanation is that melting might be a very rare process because other phenomena and mechanisms are dominant in the dynamic fault weakening.

6.1 Flash Heating and Weakening of Micro-asperity contacts

A sufficiently fast slip means that the significantly heated zone is thin (relative to the contact diameter). The increase in temperature leads to decrease in contact's shear strength. The thinness of the heated zone means that the capacity of the contact to support normal traction and the net area of the contact are not much affected. Because coefficient of friction is given by the ratio of the shear traction to normal traction, the friction coefficient is consequently reduced with the slip rate.

6.2 Thermal Pressurization

The thermal pressurization mechanism assumes presence of fluids (typically water) in pores, and the following relation between the shear traction τ, coefficient of friction f, normal traction σ, and pore pressure p:

$$\tau = f\,(\sigma - p)\ . \tag{6.1}$$

Frictional heating leads to volume expansion of fluids. The expansion is much larger than that of the solid cage. The volume expansion of fluids causes increase of pressure in the pore fluids, that is, pore pressure. (This happens unless a shear-induced dilatancy of the cage overwhelms the thermal expansion or the gouge is highly permeable.) Because the normal traction σ typically does not change during slip, the increase of the pore pressure p reduces the fault strength τ.

Predictions based on the thermal pressurization enabled plausible estimates of the fracture energy of earthquakes and could explain why strength loss over all but deeper part of seismogenic zone is too rapid for melting to take place.

The flash heating and thermal pressurization are considered as serious candidates for dominant processes causing the dynamic fault weakening. Several questions still remain to be answered. They include validity of relation (6.1), dilatancy of the gouge under shear traction, and effect of dilatancy and shear on the instantaneous permeability and poroelastic moduli.

6.3 Silica Gel

Friction experiments on a quartzite confirmed that at the time of deformation a thin layer coating the sliding surface was able to flow with a relatively low viscosity. Granulation within the shear zone produces fine silica particles which adsorb water to their surfaces and form a gel. The gel would consolidate into a strong, amorphous solid, if shear was stopped. However, the presence of shear continuously disrupts particle bonding so that the fluidized gel deforms at low strength. The gel can be also considered as a water-infiltrated porous medium. This raises question of its relation to the thermal pressurization. The mechanism also depends on the presence of quartzite in the zone.

6.4 Melting

The melting due to a sudden frictional heating, that is, the ultimate mechanism of thermal weakening, cannot be excluded from investigations despite the scarcity of supportive indications or evidence so far. If the melting occurs, it is very likely that it is not a simple process.

7 Conclusions

In this thesis we

- summarized basic relations for stress, strain and strain energy function in an elastic medium without and with an initial stress and strain,
- briefly characterized basic models of seismoactive fault and earthquake rupture,
- derived in detail all relations following from the application of the first law of thermodynamics to a smooth volume without fault (fracture) surface, volume intersecting fracture surface, and volume intersecting fracture surface and containing a fracture edge,
- derived in detail all basic relations for the seismic energy,
- briefly summarized the most important dynamic-weakening mechanisms.

In the process of deriving relations for the seismic energy we found one wrong negative sign (misprint or error?) in the important relation (4.4.21) in Kostrov and Das (1988). We also concluded that several opposite signs on p. 153 and 154 of Kostrov and Das (1988) are due to the inconsistent choice (by Kostrov and Das) of the normal to the surface of the auxiliary volume V_ε compared to that in the application of the first law of thermodynamics. (Details are given in Chapter 5 of this thesis.)

8 Appendix - Selected Geological Terms

The text of this appendix is based on Foster (1985).

Rocks of the crust: igneous, metamorphic, sedimentary.

Metamorphic rocks: rocks that have been changed either in texture or mineral composition by heat, pressure, traction, shear, or chemically active solutions. They can be divided into two textural groups: foliated - having a directional or layered aspect, non-foliated - homogeneous or massive rocks.

Types of metamorphism:

- Thermal or contact.
- Recrystallization under stress - new minerals grow in a preferred orientation; corresponding rocks are said to be foliated and most are gneiss and schist.
- Dynamic - breaking and grinding without much recrystallization; example - mylonite. Rocks are sheared, broken and ground near the surface when temperature and pressure are too low to cause any significant recrystallization. Commonly associated with fault zones.

Rocks in a fault zone: all gradiations between metamorphic rocks and ordinary schist; the fine-grained, ground rock in gouge is mylonite, the rocks gradiational with schist are called semischist or cataclastic schist or gneiss.

Clastic rocks: rocks composed of rock fragments or mineral grains from any type of preexisting rock.

Cataclasis: deformation of rock caused by fracture and rotation.

Tachylite: glassy (no crystal) mid-dark igneous rock.

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