COMENIUS UNIVERSITY, BRATISLAVA
FACULTY OF MATHEMATICS, PHYSICS AND INFORMATICS

PHYSICS OF SEISMIC WAVE PROPAGATION IN POROELASTIC MEDIA

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DÁVID GREGOR

COMENIUS UNIVERSITY
FACULTY OF MATHEMATICS,
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PHYSICS OF SEISMIC WAVE PROPAGATION IN POROELASTIC MEDIA

BACHELOR THESIS

Study program: 1160 Physics

Department: Department of Astronomy, Physics of the Earth and Meteorology

Advisor: Prof. RNDr. Peter Moczo, DrSc.

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DÁVID GREGOR

88778080

Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta:	Dávid Gregor
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Názov: Physics of seismic wave propagation in poroelastic media
Fyzika šírenia seizmických vĺn v poroelastických prostrediach

Cieľ: Spracovať teoretické podklady k numerickému modelovaniu šírenia seizmických vĺn v poroelastickom prostredí. Pripraviť výpočtový algoritmus vhodný na vývoj výpočtového programu.

Vedúci:	prof. RNDr. Peter Moczo, DrSc.
Katedra:	FMFI.KAFZM - Katedra astronómie, fyziky Zeme a meteorológie
Vedúci katedry:	prof. RNDr. Peter Moczo, DrSc.
Dátum zadania:	29.04.2013

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prof. RNDr. Jozef Masarik, DrSc.
garant študijného programu

....
študent

....
vedúci práce

88778080

Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

THESIS ASSIGNMENT

Name and Surname: Dávid Gregor
Study programme: Physics (Single degree study, bachelor I. deg., full time form)
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Title: Physics of seismic wave propagation in poroelastic media

Aim: Elaborate theoretical background for numerical modelling of seismic wave propagation in poroelastic medium. Prepare computational algorithm suitable for developing a corresponding computer code.

Supervisor: prof. RNDr. Peter Moczo, DrSc.
Department: FMFI.KAFZM - Department of Astronomy, Physics of the Earth and Meteorology
Head of department: prof. RNDr. Peter Moczo, DrSc.

Assigned: 29.04.2013

Approved: 02.05.2013 prof. RNDr. Jozef Masarik, DrSc.
Guarantor of Study Programme

Student Supervisor

I hereby declare I wrote this thesis by myself, only with the help of referenced literature, under the careful supervision of my thesis supervisor.

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I would like to express my gratitude to my supervisor Prof. Peter Moczo, who has supported me throughout my thesis with his patience, knowledge and useful remarks.

Most importantly, none of this could have happened without my family. Words cannot express how grateful I am to my mother for her love, kindness, encouragement and support she has shown during this year of writing my thesis. I would also like to thank all my of my friends who supported me in writing.

Abstrakt

Autor: Dávid Gregor
Názov práce: Fyzika šírenia seizmických vĺn v poroelastických prostrediach
Škola: Univerzita Komenského, Bratislava
Fakulta: Fakulta matematiky, fyziky a informatiky
Katedra: Katedra astronómie, fyziky Zeme a meterológie
Školiteľ: Prof. RNDr. Peter Moczo, DrSc.
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Predikcia seizmického pohybu na záujmovej lokalite je jednou z najdôležitejších úloh seizmológie zemetrasení vo vzťahu k spoločnosti. S výnimkou niekoľkých oblastí osídlenej časti povrchu Zeme nie je na dôležitých územiach a lokalitách dostatok dát na zistenie empirických vzťahov pre predikciu seizmického pohybu počas budúcich zemetrasení. Toto implikuje dôležitosť a nezastupiteľnosť teoretických metód a metód numerického modelovania vo vzťahu k predikcii seizmického pohybu. Presnosť a teda aj užitočnosť numerických metód záleží aj od miery realistického modelu povrchových štruktúr, najmä povrchových sedimentárnych štruktúr, ktoré spôsobujú anomálne zosilnenie a predĺženie trvania seizmického pohybu na povrchu. V niektorých prípadoch vodou nasýtených sedimentov je podmienkou realistického modelu prostredia a seizmického pohybu zahrnutie poroelasticity. Na rozdiel od jednozložkového kontinua (viskoelastického či elastoviskoplastického) je nutné explicitne zohľadniť aj prítomnosť pórov a kvapaliny v nich. Takéto prostredie je pre modelovanie seizmického pohybu relatívne zložité. V tejto bakalárskej práci sa venujeme základnej fyzike poroelastického prostredia, matematicko-fyzikálnemu popisu (pohybová rovnica a konštitučný vzťah) a napokon príprave výpočtového algoritmu a výpočtovej konečno-diferenčnej schémy pre numerické simulácie šírenia seizmických vĺn a seizmického pohybu.

Kľúčové slová: poroelastické médium, šírenie vĺn, seizmický pohyb

Abstract

Author:	Dávid Gregor
Thesis title:	Physics of seismic wave propagation in poroelastic media
University:	Comenius University, Bratislava
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Earthquake prediction and prediction of ground motion during future earthquakes at a site of interest are the most important tasks of the earthquake seismology in relation to society. With the exception of only few populated areas on the Earth, there is a drastic lack of earthquake recordings that could be used for empirical prediction of the earthquake ground motion. This implies importance and irreplaceability of theoretical and numerical-modelling methods with respect to prediction of the seismic ground motion during future earthquakes. Accuracy and efficiency of numerical methods depends on how realistic is a computational model, especially in case of the sediment-filled structures, which can produce anomalously amplified or prolongated earthquake motion at the Earth's surface. In case of water-saturated sediments we need to adapt poroelasticity into computational model in order to make simulation of earthquake motion more realistic. Contrary to a single-phase continuum (viscoelastic or elastoviscoplastic) it is necessary to consider presence of fluid-filled pores. This kind of material is for earthquake motion modelling relatively very difficult. In this bachelor thesis we present basic concepts of physics of poroelastic medium, derivation of equation of motion and constitutive relation, and a finite-difference scheme for numerical simulation of seismic wave propagation and seismic motion in poroelastic medium.

Key words: poroelastic media, wave propagation, seismic motion

Foreword

Seismology is the solid Earth geophysical discipline with highest societal impact, both in assessing and reducing the danger from natural hazards. It's major goal is study of earthquakes and the propagation of elastic waves through the Earth. In particular, the seismic waves and their analysis make it possible to investigate Earth's deep interior, where direct observations are impossible. However, the main task of modern seismology is directly concerned with seeking ways to reduce destructive impacts of seismic waves on human population and predict their behavior at a site of interest.

My first experience with seismology at the university was during one-semester long course of "Mechanics of Continuum". After one semester of studying I was captivated with seismological theory. How it could be on one hand a difficult mathematical subject and on the other hand able to present fascinating theoretical problems involving analysis of elastic wave propagation in complex media. This is also the reason, why the topic of my thesis is related to the earthquake prediction and prediction of ground motion. Elaboration of topic "Wave propagation in poroelastic media" gave me an opportunity to become familiar with works of one the best seismologist in field of numerical modelling of earthquake motion. It should be noted that general porous material is an anelastic, anisotropic, multi-phase medium, with all attributes of realistic model of the Earth material, and therefore implementation of this rheology for numerical modelling cannot be excluded.

Content

1 Introduction

Wave propagation in fluid-saturated porous media is a topic of increasing interest in many geophysical fields. Wave forms and amplitudes provide knowledge about the properties of the material in the subsurface and can be used in earthquake engineering, geomechanics, petroleum engineering and hydrogeology. The study of wave propagation in porous media helps us to better understand behavior of seismic waves. The term poroelasticity was first established by J. Geertsma in his work: "Problems of rock mechanics in petroleum production engineering". He defines a saturated porous material as medium formed by two interpenetrated phases. One of them is the solid phase, which constitutes the matrix of the poroelastic material, and second one is the liquid phase, which constitutes the saturated fluid. Two basic phenomena underlie behavior of poroelastic material:

- Solid to fluid coupling occurs when a change in applied stress produces a change in fluid pressure or fluid mass.
- Fluid to solid coupling occurs when a change in fluid pressure or fluid mass produces a change in the volume of the porous material.

Poroelastic behavior can explain an initially unexpected connection between causal event and its subsequent effect. Here are two historical examples:

- Water Levels Change in Well as Trains Pass. F. H. King (1892) of the University of Wisconsin reported that water levels in a well near the train station at Whitewater, Wisconsin, went up as a train approached and went down as a train left the station. The water level fluctuation was greater for a heavy freight train than for a lighter and faster passenger train (Wang 2000).
- Water Levels in Boardwalk Wells Fluctuate with Ocean Tides. In 1902th United States Geological Survey reported that water-level oscillations in wells in Atlantic City, New Jersey, were synchronous with ocean depths, because the weight of sea water at high tide compressed the underlying rock, thereby forcing pore water up the wells (Wang 2000).

Our analysis of poroelastic theory and presentation of derivation of equations of motion in poroelastic media is mainly based on books of Bourbié (1987), de la Puente (2008) and article of Biot (1956). In dealing with problem of wave propagation in porous material for dynamic analysis of the subsurface according to Bourbié (1987), two approaches are possible:

- The first approach is based on homogenization procedure, which helps us to pass from laws on microscopic scale to macroscopic ones. The microscopic laws apply here at the scale of the heterogeneity (in our case a porosity), whereas macroscopic laws refer to a scale related to the heterogeneous medium, which is in fact the measurement scale.

  We shall note that there are two homogenization methods. The first one is based on averaging procedure, where microscopic problem is first resolved at the level of an elementary cell containing an isolated heterogeneity (in our case a fluid-filled channel). From the solution to this elementary problem, we then derive the mean value on the cell of the quantity analyzed (stresses, strains...) as a function of the macroscopic value imposed at the cell boundary (strains, stresses...). After this procedure, the heterogeneous medium can be replaced by a fictitious homogenous medium. The response of the medium to an imposed force is the mean value previously calculated. The function linking them depends spatially on the geometric and mechanical parameters of the heterogeneities existing in actual medium. This method can be used for low and medium concentrations of heterogeneities, where cell-to-cell interaction processes can be ignored.

  The second homogenization method assumes the periodically repeated microscopic heterogeneous structures. If we make a spatial period tend towards zero with respect to the macroscopic scale (small parameter asymptotic method), the form of the macroscopic laws can be derived.
- The second approach relies on concepts of mechanics of continuum (existence of potential, kinetic energy and stationary principles) and deliberately ignoring the microscopic level. This method can be straightforward applied to measurable macroscopic values. This older approach is also presented by Biot (1956) in his work. The porous material presented in this thesis is based on the conceptual model of a coherent solid skeleton and a freely moving pore fluid.

In this study we will apply the second approach.

The main difference between the wave fields in a poroelastic material and those in an elastic one is the existence of a wave of the second kind, in addition to the standard compressional and shear waves. This wave, also called slow P wave, which behaves diffusively at low frequencies (e.g. seismic frequencies) and propagates at a very slow speed through the medium. This is caused by dominance of the fluid-viscosity effects over the inertial effects. As a consequence, this wave is significant only very close to the source or near material heterogeneities.

Analytical solutions for wave propagation problems in poroelastic media exist, but are usually limited to extremely simple model problems. Therefore, many studies consider the numerical solution of Biot's equations. The finite-difference method has been one of the early methods applied for this purpose in two dimensions and three dimensions.

1.1 Assumptions

- The first assumption states that the wavelength is significantly larger than largest dimension of the pores. This assumption is normally always satisfied in geophysical applications.
- The second assumption demands small fluid and solid displacements. This assumption is fully justified, because the strains in seismic studies (laboratory or field) are less than 10^{-6}.
- The third assumption requires the liquid and solid phases to be continuous.
- The fourth assumption concerns the matrix (frame) which is in this case elastic and isotropic, but it should be noted that the theory can be extended to the anisotropic elastic case.
- The fifth deals with distribution of individual phases in porous media. We assume, that all pores are interconnected (completely filled by fluid represented as continuum, so that V_f is also the volume of void space). In fact (Bourbié 1987), every natural porous medium possesses both types of porosity (disconnected and connected), so that the liquid that participates in the motion of the slow wave is merely the fraction of liquid contained in the connected porosity. It is important to realize that the coarse image by which one considers that, among the two compressional waves, one moves within the liquid and second in the solid, is false. In fact, the porous medium is a material constituted of solid and liquid phase coupled together. A more accurate image can be represented by sample as a system of two springs with eigen vibrations in of phase and out of phase.
- The final one concerns the absence of thermo-mechanical and chemical effects.

To describe the mechanics of the poroelastic material, we must define two fluid equivalents to the solid matrix stresses σ_{ij}^{m} and strains ε_{ij}^{m}, which are the fluid's pressure p and the fluid strains ε_{ij}^{f}. Strains tensors for fluid and solid can be subsequently expressed as

$\varepsilon_{ij}^{m} = \frac{1}{2}\left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)$ and $\varepsilon_{ij}^{f} = \frac{1}{2}\left(\partial U_i/\partial x_j + \partial U_j/\partial x_i\right)$, where u_i and U_i is solid and fluid displacement. Together they form $w_i = \phi\left(U_i - u_i\right)$, which is displacement vector of fluid relative to that of the solid. It is also convenient to establish relation for density of bulk as $\rho = (1-\phi)\rho_s + \phi\,\rho_f$. A poroelastic material can be described using measurable quantities from solid, fluid and matrix (frame), where matrix corresponds to skeleton part of material (poroelastic material without fluid). These quantities are denoted by letters s, m, u respectively S, M, U and summarized as follows:

- **Solid**
  - ρ_s - density of solid phase
  - K_S - bulk modulus of solid phase
  - λ - Lame's elastic coefficient
- **Fluid**
  - ρ_f - density of fluid phase
  - K_F - bulk modulus of fluid phase
  - η - viscosity
  - p - pore pressure
  - ς - variation of fluid content (the increment of fluid volume per unit volume of solid)
- **Matrix**
  - K_M - drained bulk modulus (bulk modulus of matrix)
  - λ_M - drained Lame's coefficient
  - G - matrix's shear modulus
  - ϕ - porosity
  - κ - permeability
  - T - tortuosity
- **Parameters of solid/fluid interaction**
  - K_U - undrained bulk modulus
  - M - Biot's modulus
  - α - Biot-Willis's effective stress coefficient
  - R, Q - material constants for fluid/solid interaction

B - Skempton's coefficient

Most of these quantities are well known from fundamental physics and elastic mechanics, except tortuosity, porosity and permeability. The tortuosity T is related to the ratio between the minimum (straight) and actual distance between two points of the pore space, due to the "tortuous" path of the pore connection. The porosity ϕ is defined as a ratio V_P / V_T , where V_P is the volume that takes the pore space (in our case of full saturation $V_P = V_F$) and $V_T = V_S + V_F$ is the total volume of the material. The permeability κ is a measure of the ability of porous material to transmit fluids. Other quantities will be defined subsequently in text.

2 Undrained, drained and unjacketed conditions

In this section we will discuss conditions that can be porous media exposed to, presented by Detournay (1993), de la Puente (2008) and according to website www.environment.uwe.ac.uk (2003). These conditions, specifically tests that can be carried out and characterize individual conditions are used to determine some of the fundamental coefficients, which are then employed in calculus. Three types of tests are usually executed to determine the poroelastic parameters:

- the drained test where the confining pressure p^{ext} is applied and pore pressure p remains constant
- the unjacketed test characterized by an equal increase of the confining pressure p^{ext} and pore pressure p
- the undrained test where a confining pressure p^{ext} is applied on the rock, but no fluid is allowed to enter or leave the core sample.

All these tests can be carried out in laboratory to measure mentioned parameters. An apparatus can be schematically described as pressure vessel with jacketed core of rock placed between two endcaps. Confining pressure can be applied hydraulically.

Pressure vessel's endcaps are either designed with drainage holes, in order to control of the pore pressure during exchange of fluid with the sample for the drained test, or solid for the undrained test. Note that the increments of pressure used in these experiments are typically of order of a few MPa .

2.1 Drained conditions

Drained conditions correspond to deformation at sufficiently slow rate, at fixed hydrostatic pressure p^{ext}, with the fluid being allowed to flow in or out of the deforming element however is required to keep pore pressure p constant. The state of constant pore pressure can be reached by inserting a tube into the rock and connecting it to a fluid reservoir at the same pressure. The parameter K_M can be obtained by measuring the volumetric strain due to changes in applied stress while holding pore pressure constant. In this case, all the external pressure is transmitted to the frame, and therefore one can define the drained bulk modulus of the matrix as $K_M = -p^{\text{ext}}/\varepsilon_{kk}^m$. Situation describing drained conditions is pictured in Figure 2.1a). The representative elementary volume of porous material under drained conditions is pictured in Figure 2.1b).

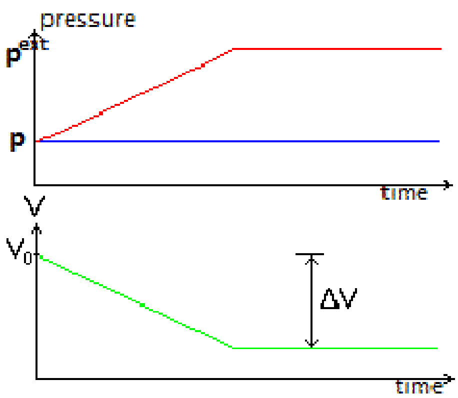

Figure 2.1a) Time dependence of pressure and volume under drained conditions.

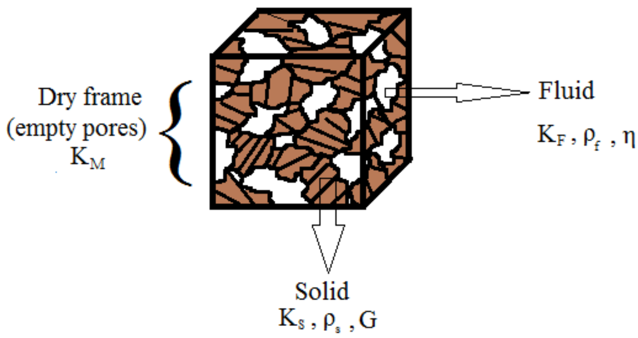

Figure 2.1b) The representative elementary volume of porous material under drained conditions.

2.2 Undrained conditions

The opposite limit is undrained deformation occurring at the time scale that is too short, so the fluid is not allowed to flow in or out during deformation and, in general, changes of pore pressure p are induced. This means $\varsigma = 0$. After some period of time consolidation will occur, which is dissipation of excess pore pressure, accompanied by volume change after opening endcaps. The rate of consolidation is dependent on the permeability of the solid phase and size of the consolidating layer. Transient undrained conditions prevail during consolidation, but eventually, when all of the excess pore pressure has been dissipated, conditions are the same as those for drained case. Two measurements can be made: the volumetric change ΔV and confining pressure change Δp^{ext} for the determination of undrained modulus $K_U = V\Delta p^{\text{ext}}/\Delta V$ and the pore pressure change Δp for B $\left(B = \Delta p/\Delta p^{\text{ext}}\right)$, which is Skempton's coefficient. Situation describing drained conditions is pictured in Figure 2.2a). The representative elementary volume of porous material under undrained conditions is pictured in Figure 2.2b).

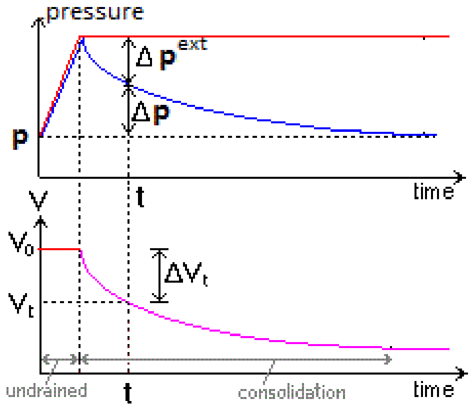

Figure 2.2a) Time dependence of pressure and volume under undrained conditions.

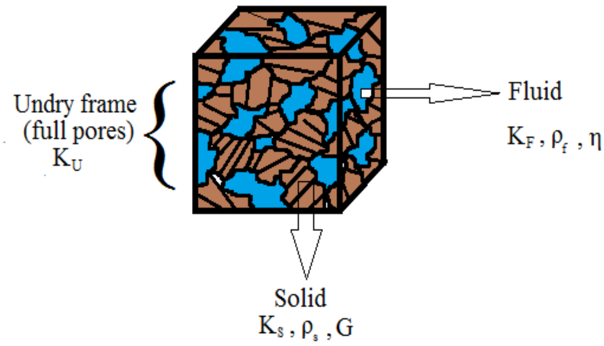

Figure 2.2b) The representative elementary volume of porous material under undrained conditions.

2.3 Unjacketed conditions

It is related to the case when the increase in confining pressure is equal to the increase in pore fluid pressure. This test can be carried out by immersing whole poroelastic sample in fluid so the pressure p is applied. This pressure will distribute itself among the $1-\phi$ part of the frame and the ϕ fluid part of the surface of the material. From this experiment we will obtain two relations $K_S = -p/\varepsilon_{kk}^m$ and $K_F = -p/\varepsilon_{kk}^f$ which we will be subsequently used in the next chapter.

3 Constitutive equations

In this chapter we will present derivation of constitutive equations for poroelastic media following work of de la Puente (2008). The most general form of the constitutive equation for a fluid-filled porous material, is given by

$$\begin{aligned} \sigma_{ij}^m &= Q\,\varepsilon_{kk}^f\,\delta_{ij} + \left(K - \frac{2}{3}G\right)\varepsilon_{kk}^m\,\delta_{ij} + 2G\,\varepsilon_{ij}^m \\ \sigma^f &= R\,\varepsilon_{kk}^f + Q\,\varepsilon_{kk}^m \end{aligned} \tag{3.1}$$

where $\sigma^f = -\phi\, p$. This relation is called partial stress formulation. The first equation of (3.1) for stress in terms of strain and pore pressure may be inverted to solve for strain, leading to:

$$\varepsilon_{ij}^m = \frac{1}{2G}\sigma_{ij}^m + \left[\frac{1}{9\left(K - Q^2/R\right)} - \frac{1}{6G}\right]\delta_{ij}\,\sigma_{kk}^m \tag{3.2}$$

Parameter G can be easy obtained by subjecting the material described in (3.1) to a pure shear deformation, so that $\varepsilon_{ij}^m = \varepsilon_{ij}^f = 0$ for $i = j$. It can then be shown, that $\sigma_{ij}^m = 2\,G\,\varepsilon_{ij}^m$, so the parameter G responds to matrix's shear modulus $G = \mu_M$. For identifying parameters K, Q, R we will use conditions described in chapter 2.

Using (3.1) under the drained conditions one can obtain:

$$\begin{aligned} -p^{\text{ext}} &= K\,\varepsilon_{kk}^m\,\delta_{ij} + Q\,\varepsilon_{kk}^f\,\delta_{ij} \\ 0 &= Q\,\varepsilon_{kk}^m + R\,\varepsilon_{kk}^f \end{aligned} \tag{3.3}$$

After applying $K_M = -p^{\text{ext}}/\varepsilon_{kk}^m$ one can obtain a relation between K_M and the still unknown poroelastic parameters K, Q and R.

$$K_M = K - \frac{Q^2}{R} \tag{3.4}$$

After carrying out unjacketed experiment we mentioned earlier, equation (3.1) becomes:

$$\begin{aligned} -(1-\phi)\,p &= K\,\varepsilon_{kk}^m\,\delta_{ij} + Q\,\varepsilon_{kk}^f\,\delta_{ij} \\ -\phi p &= Q\,\varepsilon_{kk}^m + R\,\varepsilon_{kk}^f \end{aligned} \tag{3.5}$$

We can see that the pressure is acting from the inside of the porous rock, and therefore the compressional properties deduced from the experiment are those of the rock or solid instead of those of matrix. One can use $K_S = -p/\varepsilon_{kk}^m$ and $K_F = -p/\varepsilon_{kk}^f$ introduced earlier to obtain further set of constraints on the unknown parameters K, Q and R as follows

$$\begin{aligned} 1-\phi &= \frac{Q}{K_F} + \frac{K}{K_S} \\ \phi &= \frac{R}{K_F} + \frac{Q}{K_S} \end{aligned} \tag{3.6}$$

which combined with (3.5), builds up a system of three equations and unknown parameters which can be solved as follows:

$$K = \frac{(1-\phi)(1-\phi - K_M/K_S)K_S + \phi K_S K_M/K_F}{1-\phi - K_M/K_S + \phi K_S/K_F}$$
$$Q = \frac{\phi(1-\phi - K_M/K_S)K_S}{1-\phi - K_M/K_S + \phi K_S/K_F} \quad (3.7)$$
$$R = \frac{\phi^2 K_S}{1-\phi - K_M/K_S + \phi K_S/K_F}$$

Using relations above, we can introduce some new parameters and following relations can be found

$$K = M \quad (\alpha - \phi)^2 + K_M$$
$$Q = M\,\phi(\alpha - \phi)$$
$$R = M\,\phi^2 \quad (3.8)$$
$$B = \frac{\alpha M}{K_M + \alpha^2 M}$$

$$K_U = K_M + \alpha^2 M \quad (3.9)$$

where the fluid-solid coupling Biot's modulus M is denoted as:

$$M = \frac{K_S}{1-\phi - K_M/K_S + \phi K_S/K_F} \quad (3.10)$$

and the effective Biot-Willis's stress coefficient α has this form:

$$\alpha = \frac{\phi(Q+R)}{R} = 1 - \frac{K_M}{K_S} \quad (3.11)$$

From equations (3.9, 3.10, 3.11) one can obtain:

$$\frac{1}{M} = \frac{\alpha - \phi}{K_S} + \frac{\phi}{K_F} \quad (3.12)$$

$$K_U = \frac{\phi\left(\frac{1}{K_S} - \frac{1}{K_F}\right) + \left(\frac{1}{K_S} - \frac{1}{K_M}\right)}{\frac{1}{K_M}\phi\left(\frac{1}{K_S} - \frac{1}{K_F}\right) + \frac{1}{K_S}\left(\frac{1}{K_S} - \frac{1}{K_M}\right)} \quad (3.13)$$

Note that (3.11, 3.12, 3.13) are quite compatible with limit cases. For solid medium corresponding to $\alpha = \phi = 0$, the expected values $K_M = K_U = K_S$ and $M \to \infty$ are obtained. For fluid medium corresponding to $\alpha = \phi = 1$, we obtain $K_M = 0$ and $K_F = K_U = M$.

After defining new constants, we can express relation (3.1) as:

$$\sigma_{\mathrm{ij}}^{m} = \left[(\alpha-\phi)^{2} M\, \varepsilon_{kk}^{m} + (\alpha-\phi) M\, \phi\, \varepsilon_{kk}^{f}\right]\delta_{\mathrm{ij}} + \left(K_{M}-\frac{2}{3}G\right)\delta_{ij}\,\varepsilon_{kk}^{m} + 2G\,\varepsilon_{ij}^{m}$$
$$\sigma^{f} = (\alpha-\phi) M\, \phi\, \varepsilon_{kk}^{m} + M\, \phi^{2} \varepsilon_{kk}^{f} \qquad (3.14)$$

First expression of (3.14) can be further simplified by adapting elastic tensor from Hooke's law, which has form:

$$c_{ijkl}^{m} = \left(K_{M}-\frac{2}{3}G\right)\delta_{ij}\,\delta_{kl} + G\left(\delta_{ik}\,\delta_{jl}+\delta_{il}\,\delta_{jk}\right) \qquad (3.15)$$

so one can obtain:

$$\sigma_{\mathrm{ij}}^{m} = \left[(\alpha-\phi)^{2} M\, \varepsilon_{kk}^{m} + (\alpha-\phi) M\, \phi\, \varepsilon_{kk}^{f}\right]\delta_{\mathrm{ij}} + c_{\mathrm{ijkl}}^{m}\,\varepsilon_{kk}^{m}$$
$$\sigma^{f} = (\alpha-\phi)\, M\, \phi\, \varepsilon_{kk}^{m} + M\phi^{2}\, \varepsilon_{kk}^{f} \qquad (3.16)$$

Where $\sigma_{ij}^{\mathrm{eff}} = c_{ijkl}^{m}\,\varepsilon_{kl}^{m}$ can be denoted as effective stress.

Now we can express equations (3.16) in terms of the variation of fluid content $\varsigma = -\phi\left(\varepsilon_{kk}^{f}-\varepsilon_{kk}^{m}\right)$ and the total stress $\sigma_{ij} = \sigma_{ij}^{m} + \sigma^{f}\,\delta_{ij}$, thus obtaining

$$\sigma_{ij} = \sigma_{ij}^{\mathrm{eff}} - \alpha\, p\, \delta_{ij}$$
$$p = M\left(\varsigma-\alpha\,\varepsilon_{kk}^{m}\right) \qquad (3.17)$$

which is final form of the poroelastic constitutive laws for isotropic case.

Relation (3.17) can be modified using w, where w can be expressed by means of variation of fluid content $\varsigma = -\partial_{i}\, w_{i}$, which then gives:

$$\sigma_{ij} = 2G\,\varepsilon_{ij}^{m} + \left(K_{M}-\frac{2}{3}G\right)\varepsilon_{kk}^{m}\,\delta_{ij} - \alpha\,\delta_{ij}\, p$$
$$-p = M\left(\partial_{i}\, w_{i}+\alpha\,\varepsilon_{kk}^{m}\right) \qquad (3.18)$$

where $K_{M}-\frac{2}{3}G$ is equal to drained Lame's coefficient λ_{M}.

Partial summary: Relation (3.17) represents isotropic constitutive equations for porous media.

4 Equation of motion

To obtain equation of motion we must at first define density of kinetic and potential energy together with dissipation energy, which afterwards insert into the Lagrangian equation, following work of Bourbié (1987).

4.1 Dissipation

Processes of dissipation in this chapter are assumed to result only from the relative movement of solid and fluid. In the neighborhood of equilibrium, vector $\dot{w}_i$ and the dissipative force X_j are linked by a linear equation such as:

$$\dot{w}_i = b_{ij}^{-1} X_j \tag{4.1}$$

Based on (4.1), a dissipation pseudo-potential $\mathbb{D}$ can be introduced, which is a positive definite quadratic form of representative matrix b_{ij}, such that:

$$\mathbb{D} = \frac{1}{2} \dot{w}_i \, b_{ij} \, \dot{w}_i^T \tag{4.2}$$

For isotropic porous material the off-diagonal elements in the b_{ij} tensor are zero for $i \neq j$ and the diagonal elements are identical, so the common form is obtained:

$$\mathbb{D} = \frac{1}{2} b_{ij} \, \dot{w}_i^2 \tag{4.3}$$

We can present derivation of classic Darcy's law (in steady-state conditions) from Navier-Stokes microscopic equations for fluid of viscosity η, by making an analogy with Poiseuille's law. This gives rise to the expression of the hydraulic permeability in the form

$$b = \frac{\eta}{\kappa} \tag{4.4}$$

where κ, the absolute permeability, depends only on geometry of the porous media. Considering flow with at sufficiently low velocity we can write Darcy's law as:

$$X_i = \frac{\eta}{\kappa} \dot{w}_i \tag{4.5}$$

4.2 Kinetic energy

The kinetic energy C of the system per unit volume may be expressed as:

$$2C = \rho_u \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} + 2\rho_{uw} \frac{\partial u_i}{\partial t} \frac{\partial w_i}{\partial t} + \rho_w \frac{\partial w_i}{\partial t} \frac{\partial w_i}{\partial t} \tag{4.6}$$

This expression is based on the assumption that the material statistically isotropic. The specific case when the relative movement between fluid and solid is completely prevented in some way (i.e. $\dot{w}_i = 0$), serves to identify ρ_u in equation (4.6) as:

$$\rho_u = \rho = (1 - \phi) \rho_s + \phi \rho_f \tag{4.7}$$

The terms ρ_{uw} and ρ_w will be identified subsequently. Moreover, after defining new parameters ρ_{12} in chapter "Propagation of P and S wave" we will be able to rewrite (4.6) in these terms as

$$C = \frac{1}{2}(1-\phi)\,\rho_s \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} + \frac{1}{2}\phi\,\rho_f \frac{\partial U_i}{\partial t}\frac{\partial U_i}{\partial t} - \frac{1}{2}\rho_{12}\left(\frac{\partial U_i}{\partial t} - \frac{\partial u_i}{\partial t}\right)\left(\frac{\partial U_i}{\partial t} - \frac{\partial u_i}{\partial t}\right) \tag{4.8}$$

where kinetic energy is now expressed as the sum of the kinetic energy density of the solid, kinetic energy density of the fluid and the kinetic energy density due to the relative motion of an additional apparent fluid mass arising from the inertial drag of the fluid.

4.3 Potential energy

If we assume only small disturbances, the expression of U can be limited to the quadratic terms. The assumptions of isotropy implies that this expression involves the first two invariants of strain tensor ε_{kk}^m and ε_{ij}^m, as well variation of fluid content ς. One can therefore write:

$$2U = \left(\lambda_M + \alpha^2 M\right)\varepsilon_{kk}^m\,\varepsilon_{kk}^m + 2G\,\varepsilon_{ij}^m\,\varepsilon_{ij}^m - 2\alpha M\,\varepsilon_{kk}^m\,\varsigma + M\,\varsigma^2 \tag{4.9}$$

For $\alpha = \varsigma = 0$, the single-phase case can be obtained.

4.4 Hamilton's principle

The dynamic behavior of homogenous system in space, including continua, can be specified by a single function, a Lagrangian density $\mathcal{L}$, which is a function of, say, n local dependent variables $q_1, q_2, \ldots\ldots q_n$ and their first derivatives $\dot{q}_i, q_{i,j}$. Generally there is no direct dependence of $\mathcal{L}$ on the independent variables x_j and t; there is only an indirect dependence since $q_i, \dot{q}_i, q_{i,j}$ are functions of t and x_j. In our case we have to add component responsible for dissipative potential $\mathbb{D}$ depending on $\dot{q}_i$.

Hamilton's principle (for example see Achenbach 1975) states that of all paths of motion between two instants t_1 and t_2, the actual path taken by the system is such that integral over time and space of the Lagrangian density $\mathcal{L}$ and of the work of the dissipated forces is stationary.

If we use Lagrangian equation

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial}{\partial x_j}\frac{\partial \mathcal{L}}{\partial q_{i,j}} - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathbb{D}}{\partial \dot{q}_i} = 0 \tag{4.10}$$

where $q_i = u_i$ or w_i in our case. The application of (4.10) by using expressions (4.3), (4.6) and (4.9) for isotropic case, leads to equation of motion:

$$\begin{aligned} \sigma_{ij,j} &= \rho\, \ddot{u}_i + \rho_{uw}\, \ddot{w}_i \\ -p_{,i} &= \rho_{uw}\, \ddot{u}_i + \rho_w\, \ddot{w}_i + b\, \dot{w}_i \end{aligned} \tag{4.11}$$

Relation (4.11) represents equations of motion. To identify still unknown parameters ρ_{uw} and ρ_w we can use several assumptions.

If there is no relative fluid movement with respect to the solid movement (i.e. $w_i = 0$), the first equation is reduced to the equation of movement in solid, while the second is reduced to the equation of movement of a fluid, thus $\rho_{uw} = \rho_f$. Now if the fluid is the rest $\left(w_i = -\phi\, u_i\right)$, the second equation is reduced to:

$$-p_{,i} = \left(\rho_f - \rho_w\, \phi\right) \ddot{u}_i - \phi\, b\, \dot{u}_i \tag{4.12}$$

This shows that, if overall acceleration occurs, a force must be exerted on the fluid to prevent average displacement. For its inertial part, this coupling force is:

$$\left(\rho_f - \rho_w\, \phi\right) \ddot{u}_i \tag{4.13}$$

To describe this coupling effect, similar to the mass effect added in the analysis of the movement of an obstacle in a fluid, it is usual to establish parameter tortuosity T, such that:

$$\rho_w = \frac{T}{\phi}\rho_f \tag{4.14}$$

The tortuosity T was introduced in chapter one and it is related not only to porosity but also to the geometry of the medium where the flow occurs. If T tends toward 1 as ϕ tends towards 1, the medium will be reduced to a fluid. We have to point out, that the way in which this limiting case is reached does not only depend upon the way ϕ tends towards 1, but especially upon the geometry of the porous medium. The tortuosity is determined by Berryman's (1980) method using

$$T = 1 - r\left(1 - 1/\phi\right) \tag{4.15}$$

where r is a factor to be evaluated from a microscopic model of the geometry of the frame. For the case of solid spherical grains $r = \frac{1}{2}$.

Equations (4.11) are thus written as:

$$\begin{aligned} \sigma_{ij,j} &= \rho \ \ddot{u}_i + \quad \rho_f \ \ddot{w}_i \\ -p_{,i} &= \rho_f \ \ddot{u}_i + \frac{T}{\phi} \rho_f \ \ddot{w}_i + b \, \dot{w}_i \end{aligned} \tag{4.16}$$

Relations for pore pressure p and total stress σ_{ij} in equation (4.17) are from chapter three denoted as:

$$\begin{aligned} \sigma_{ij} &= c^m_{ijkl} \, \varepsilon^m_{kl} - \alpha \, p \, \delta_{ij} \\ p &= M \left(\varsigma - \alpha \, \varepsilon^m_{kk} \right) \end{aligned} \tag{4.17}$$

Make a partial summary: Equations of motion (4.17) and constitutive equations (4.18) describe isotropic poroelastic wave propagation at low frequencies.

5 Propagation of S and P wave

In this chapter we will analyze wave propagation of S and P waves following works of Biot (1956) and Bourbié (1987), using equations of motion (4.16).

The introduction of the equations of stresses as a function of displacement u_i a U_i in equation (4.16) yields the equations of motion in the form:

$$\begin{aligned} &(\lambda + 2G) \partial_j \partial_i u_i + \Upsilon \partial_j \partial_i U_i - G \, \varepsilon_{ijk} \, \partial_j \, \varepsilon_{klm} \, \partial_l \, u_m = \rho_{11} \, \ddot{u}_i \\ &+ \rho_{12} \, \ddot{U}_i + \beta \left(\dot{u}_i - \dot{U}_i \right) \end{aligned} \tag{5.1}$$

$$\Upsilon \partial_j \partial_i u_i + R \partial_j \partial_i U_i = \rho_{12} \, \ddot{u}_i + \rho_{22} \, \ddot{U}_i - \beta \left(\dot{u}_i - \dot{U}_i \right) \tag{5.2}$$

in which we noted:

$$\begin{aligned} \lambda_0 &= \lambda_M + M (\alpha - \phi)^2 \\ \Upsilon &= \qquad M \phi (\alpha - \phi) \\ R &= \qquad M \phi^2 \\ \rho_{11} &= \rho - \phi \, \rho_f \, (2 - T) \\ \rho_{12} &= \qquad \phi \, \rho_f \, (1 - T) \\ \rho_{22} &= \qquad \phi \, \rho_f \, T \\ \beta &= \frac{\eta \, \phi^2}{\kappa} \end{aligned} \tag{5.3}$$

Note that (5.1, 5.2) can be written for the limit cases of the perfect fluid and the solid. In fact, if the medium is a perfect fluid, the set of parameters to be considered is $\phi = \alpha = T = 1$ and $G = \beta = 0 \ (\kappa \to \infty, \ \eta = 0)$. Equation (5.1) disappears and equation (5.2) gives the dynamic equation of perfect fluid under the assumption of small displacement:

$$M \, \partial_j \, \partial_i \, u_i \; = \; \rho_f \, \ddot{U}_i \tag{5.4}$$

It should be noted that equation (5.1) disappear naturally and not by assuming $U_i \; = \; u_i$.

Now, if the medium consists only of solid phase, letting $\alpha \; = \; \phi \; = \; 0, \; \lambda_M \; = \; \lambda, \; T \; \rightarrow \; \infty$, Eq. (5.2) then gives

$$0 \; = \; -T \, \phi \, \rho_f \, \ddot{u}_i \; + \; T \, \phi \, \rho_f \, \ddot{U}_i \; - \; \beta \left(\dot{u}_i - \dot{U}_i \right) \tag{5.5}$$

while equation (5.1) gives:

$$\begin{aligned} (\lambda + 2\,G) \partial_j \, \partial_i \, u_i \; - \; G \, \varepsilon_{ijk} \, \partial_j \, \varepsilon_{klm} \, \partial_l \, u_m \; - \; \rho_m \, \ddot{u}_i \; = \; T \, \phi \, \rho_f \, \ddot{u}_i \\ - \; T \, \phi \, \rho_f \, \ddot{U}_i \; + \; \beta \left(\dot{u}_i - \dot{U}_i \right) \end{aligned} \tag{5.6}$$

Equation (5.6) is clearly the equation of the dynamics of elastic solids, since the second member is cancelled owing to (5.5).

5.1.1 Existence of S wave

Let us first examine the case without dissipation $\left(\beta \; = \; 0 \right)$ for the shear wave (S wave) or isovolumetric wave $\left(\partial_i \, u_i \; = \; \partial_i \, U_i \; = \; 0 \right)$ such that:

$$\begin{aligned} u_i \; &= \; \varepsilon_{ijk} \, \partial_j \, \Lambda_k^1 \\ U_i \; &= \; \varepsilon_{ijk} \, \partial_j \, \Lambda_k^2 \end{aligned} \tag{5.7}$$

Equation (5.7) introduced into (5.1, 5.2) becomes

$$\begin{aligned} \partial_i \, \partial_j \, \Lambda_j^1 \; - \; \frac{1}{V_I^2} \ddot{\Lambda}_i^1 \; = \; 0 \\ -\left(\frac{\rho_{12}}{\rho_{22}} \right) \ddot{\Lambda}_i^1 \; = \; \ddot{\Lambda}_i^2 \end{aligned} \tag{5.8}$$

where velocity V_I is given by:

$$V_I \; = \; \left(\frac{G}{\rho_{11} - \dfrac{\rho_{12}^2}{\rho_{22}}} \right)^{\frac{1}{2}} \tag{5.9}$$

Since the fluid does not respond to the shear forces, it only influences the shear wave through inertial effects.

5.1.2 Existence of P waves

Now let us examine the dilatational waves without dissipation $(\beta = 0)$ and such that:

$$\begin{aligned} u_i &= \partial_i \Phi^1 \\ U_i &= \partial_i \Phi^2 \end{aligned} \tag{5.10}$$

These waves correspond to dilatational waves (P waves) that are irrotational $\left(\varepsilon_{ijk}\, \partial_j\, \varepsilon_{klm}\, \partial_l\, u_m = \varepsilon_{ijk}\, \partial_j\, \varepsilon_{klm}\, \partial_l\, U_m = 0\right)$. By introducing (5.10) in (5.1, 5.2) with $(\beta = 0)$, the following equations are obtained:

$$\begin{aligned} (\lambda + 2G)\partial_j \partial_j \Phi^1 + \Upsilon \partial_j \partial_j \Phi^2 - \rho_{11} \ddot{\Phi}^1 - \rho_{12} \ddot{\Phi}^2 &= 0 \\ \Upsilon \partial_j \partial_j \Phi^1 + R \partial_j \partial_j \Phi^2 - \rho_{12} \ddot{\Phi}^1 - \rho_{22} \ddot{\Phi}^2 &= 0 \end{aligned} \tag{5.11}$$

It is convenient to establish reference velocity V_D, defined by

$$\mathrm{V}_D^2 = \frac{\lambda + 2G + \mathrm{R} + 2\Upsilon}{\rho} \tag{5.12}$$

where

$$\rho = \rho_{11} + 2\rho_{12} + \rho_{22} \tag{5.13}$$

Velocity V_D represents dilatational wave in (5.11) under the condition $u_i = U_i$. It is convenient introduce non-dimensional parameters:

$$\begin{aligned} &\gamma_{11} = \frac{\rho_{11}}{\rho}, \qquad \gamma_{22} = \frac{\rho_{22}}{\rho}, \qquad \gamma_{12} = \frac{\rho_{12}}{\rho} \\ &\theta_{11} = \frac{\lambda + 2G}{\lambda + 2G + \mathrm{R} + 2\Upsilon}, \ \theta_{22} = \frac{\mathrm{R}}{\lambda + 2G + \mathrm{R} + 2\Upsilon}, \ \theta_{12} = \frac{\Upsilon}{\lambda + 2G + \mathrm{R} + 2\Upsilon} \end{aligned} \tag{5.14}$$

The θ_{ij}-parameters define the elastic properties of the material while the γ_{ij}-parameters define its dynamic properties. It can be shown that (5.14) parameters satisfy following identity:

$$\gamma_{11} + 2\gamma_{12} + \gamma_{22} = \theta_{11} + 2\theta_{12} + \theta_{22} = 1 \tag{5.15}$$

With these parameters equation (5.11) becomes:

$$\begin{aligned} \partial_j \partial_j \left(\theta_{11} \Phi^1 + \theta_{12} \Phi^2\right) &= \frac{1}{\mathrm{V}_D^2} \frac{\partial^2}{\partial t^2} \left(\gamma_{11} \Phi^1 + \gamma_{12} \Phi^2\right) \\ \partial_j \partial_j \left(\theta_{12} \Phi^1 + \theta_{22} \Phi^2\right) &= \frac{1}{\mathrm{V}_D^2} \frac{\partial^2}{\partial t^2} \left(\gamma_{12} \Phi^1 + \gamma_{22} \Phi^2\right) \end{aligned} \tag{5.16}$$

Solutions of these equations can be written in the form:

$$\Phi^1 = \Phi_{10} \exp\left[i\left(k\,x + \omega\,t\right)\right]$$
$$\Phi^2 = \Phi_{20} \exp\left[i\left(k\,x + \omega\,t\right)\right] \tag{5.17}$$

The velocity V of these waves can be expressed as:

$$V = \frac{\omega}{k} \tag{5.18}$$

After substituting expressions (5.17) into (5.16) and denoting parameter z as

$$z = \frac{V_D^2}{V^2} \tag{5.19}$$

we will obtain:

$$z\left(\theta_{11}\Phi_{10} + \theta_{12}\Phi_{20}\right) = \gamma_{11}\Phi_{10} + \gamma_{12}\Phi_{20}$$
$$z\left(\theta_{12}\Phi_{10} + \theta_{22}\Phi_{20}\right) = \gamma_{12}\Phi_{10} + \gamma_{22}\Phi_{20} \tag{5.20}$$

Eliminating Φ_{10} and Φ_{20} yields an equation for z:

$$\left(\theta_{11}\theta_{22} - \theta_{12}^2\right)z^2 - \left(\theta_{11}\theta_{22} + \theta_{22}\gamma_{11} - 2\theta_{12}\gamma_{12}\right)z + \left(\gamma_{11}\gamma_{22} - \gamma_{12}^2\right) = 0 \tag{5.21}$$

This equation has two roots z_1, z_2 corresponding to two velocities of propagation V_1, V_2

$$V_1^2 = \frac{V_D^2}{z_1}$$
$$V_2^2 = \frac{V_D^2}{z_2} \tag{5.22}$$

There are therefore two dilatational waves. The roots z_1, z_2 are always positive, since the matrices of coefficients γ, θ of equations (5.20) are symmetric and are associated with positive definite quadratic forms representing respectively the potential and kinetic energies.

For more detailed analysis (Biot 1956), it can be shown that one of the characteristic movements corresponds to a movement in which solid and fluid displacements are in phase, and the second to a movement in which the displacements are out of phase. The wave that corresponds to the relative movement case is known as the slow wave or the wave of the second kind. This terminology derives from the fact that the associated velocity V_1 is much lower than the velocity V_2 of the in-phase movement wave called wave of the first kind. Wave with velocity V_2 corresponds to classic P-wave, which can be noticed in the absence of fluid (Bourbié 1987).

5.1.3 Wave velocities and attenuations

As in the case of no attenuation, we may separate the equations (5.1) and (5.2) into rotational and dilatational waves, but first we must introduce characteristic frequency, denoted as:

$$\mathrm{f_c} = \frac{\beta}{2\,\pi\,\rho\left(\gamma_{12} + \gamma_{22}\right)} \tag{5.23}$$

Applying the divergence operator to (5.1, 5.2) we have the equations for dilatational waves:

$$\begin{aligned}\partial_j\,\partial_j\left[(\lambda+2\,\mu)\,\Phi^1 + \Upsilon\,\Phi^2\right] &= \frac{\partial^2}{\partial t^2}\left(\rho_{11}\,\Phi^1 + \rho_{12}\,\Phi^2\right) + \beta\frac{\partial}{\partial t}\left(\Phi^1 - \Phi^2\right)\\ \partial_j\,\partial_j\left[\qquad \Upsilon\,\Phi^1 + \mathrm{R}\,\Phi^2\right] &= \frac{\partial^2}{\partial t^2}\left(\rho_{12}\,\Phi^1 + \rho_{22}\,\Phi^2\right) - \beta\frac{\partial}{\partial t}\left(\Phi^1 - \Phi^2\right)\end{aligned} \tag{5.24}$$

Likewise, applying the curl operator, we will find the equation for rotational waves:

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\left(\rho_{11}\,\Lambda_i^1 + \rho_{12}\,\Lambda_i^2\right) + \beta\frac{\partial}{\partial t}\left(\Lambda_i^1 - \Lambda_i^2\right) &= G\,\partial_i\,\partial_j\,\Lambda_j^1\\ \frac{\partial^2}{\partial t^2}\left(\rho_{12}\,\Lambda_i^1 + \rho_{22}\,\Lambda_i^2\right) - \beta\frac{\partial}{\partial t}\left(\Lambda_i^1 - \Lambda_i^2\right) &= 0\end{aligned} \tag{5.25}$$

Let us first examine a rotational plane wave propagating in the x-direction, then solutions of equations (5.25) are written in the form:

$$\begin{aligned}\Lambda_i^1 &= \Lambda^1 \exp\left[\mathrm{i}\left(\mathrm{k\,x} + \omega\,\mathrm{t}\right)\right]\\ \Lambda_i^2 &= \Lambda^2 \exp\left[\mathrm{i}\left(\mathrm{k\,x} + \omega\,\mathrm{t}\right)\right]\end{aligned} \tag{5.26}$$

Substitution in equations (5.26) and elimination of the constants Λ^1 and Λ^2 yield the relation

$$\frac{G\,\mathrm{k}^2}{\rho\,\omega^2} = \mathrm{E_{Re}} - \mathrm{i}\,\mathrm{E_{Im}} \tag{5.27}$$

with

$$\begin{aligned}\mathrm{E_{Re}} &= \frac{1 + \gamma_{22}\dfrac{\gamma_{11}\,\gamma_{22} - \gamma_{12}^2}{\left(\gamma_{12}+\gamma_{22}\right)^2}\left(\dfrac{f}{f_\mathrm{c}}\right)^2}{1 + \left(\dfrac{\gamma_{22}}{\gamma_{12}+\gamma_{22}}\right)^2\left(\dfrac{f}{f_\mathrm{c}}\right)^2}\\ \mathrm{E_{Im}} &= \frac{\left(\gamma_{12}+\gamma_{22}\right)\left(\dfrac{f}{f_\mathrm{c}}\right)}{1 + \left(\dfrac{\gamma_{22}}{\gamma_{12}+\gamma_{22}}\right)^2\left(\dfrac{f}{f_\mathrm{c}}\right)^2}\end{aligned} \tag{5.28}$$

the frequency of the wave is $f = \omega / 2\pi$. Assuming, that k is complex

$$k = k_{Re} + i\, k_{Im} \tag{5.29}$$

indicates, that phase velocity of rotational wave has form:

$$v_r = \frac{\omega}{\left| k_{Re} \right|} \tag{5.30}$$

We can establish a reference velocity

$$V_r = \left(\frac{\mu}{\rho} \right)^{\frac{1}{2}} \tag{5.31}$$

which is the velocity of rotational waves, if there is no relative motion between fluid and solid. We derive from (5.27) using (5.28), (5.29), (5.30), that:

$$\frac{v_r}{V_r} = \frac{\sqrt{2}}{\left[\sqrt{\left(E_{Re}^2 + E_{Im}^2 \right)} + E_{Re} \right]^{\frac{1}{2}}} \tag{5.32}$$

This velocity ratio is a function only of the frequency ratio f / f_c and the dynamic parameter γ_{ij}. From relation (5.29) it is evident, that k_i is attenuation coefficient of the rotational wave. We can introduce a reference length

$$L_r = \frac{V_r}{2\pi f_c} \tag{5.33}$$

Introducing (5.28), (5.29), (5.33) into (5.27), one can obtain:

$$k_{Im} = \frac{f}{f_c} \frac{\left[\sqrt{\left(E_{Re}^2 + E_{Im}^2 \right)} - E_{Re} \right]^{\frac{1}{2}}}{\sqrt{2}\, L_r} \tag{5.34}$$

Using (5.29) and (5.26) can be shown that the amplitude of the wave as a function of the distance x is proportional to $\exp(-k_{Im}\, x)$ while the real part is related to the phase velocity v_r as we have shown earlier.

We now consider P-waves. They are governed by equation (5.24). Again we consider plane waves and express solutions of (5.2) as:

$$\begin{aligned} \Phi^1 &= \Phi_{10} \exp\left[i\left(k\, x + \omega\, t \right) \right] \\ \Phi^2 &= \Phi_{20} \exp\left[i\left(k\, x + \omega\, t \right) \right] \end{aligned} \tag{5.35}$$

Introducing the solution (5.35) into the propagation equations (5.24) and eliminating the constants Φ_{10} and Φ_{20} we obtain

$$\left[(\lambda+2\mu)-\Upsilon^2\right]\frac{k^4}{\omega^4} - \left[R\rho_{11}+(\lambda+2\mu)\rho_{22}-2\Upsilon\rho_{12}\right]\frac{k^2}{\omega^2}$$
$$+ \rho_{11}\rho_{22} - \rho_{12}^2 + \frac{i\beta}{\omega}\left\{\left[(\lambda+2\mu)+R+2\Upsilon\right]\frac{k^2}{\omega^2}-\rho\right\} = 0 \tag{5.36}$$

With the variables already presented in (5.14) this equation may be written in non-dimensional form

$$\left(\theta_{11}\theta_{22}-\theta_{12}^2\right)z^2 - \left(\theta_{22}\gamma_{11}+\theta_{11}\gamma_{22}-2\theta_{12}\gamma_{12}\right)z +$$
$$+ \left(\gamma_{11}\gamma_{22}-\gamma_{12}^2\right) + \frac{i\beta}{\omega\rho}(z-1) = 0 \tag{5.37}$$

with

$$z = \frac{k^2}{\omega^2}V_D^2 \tag{5.38}$$

In this case k is complex and therefore z is also complex. If we put $\beta = 0$ in equation (5.36) we obtain (5.21) whose roots are z_1, z_2. With these roots (5.37) might be rewritten as

$$\left(z-z_1\right)\left(z-z_2\right) + iM(z-1) = 0 \tag{5.39}$$

where

$$M = \frac{\beta}{\omega\rho\left(\theta_{11}\theta_{22}-\theta_{12}^2\right)} \tag{5.40}$$

The roots of equation (5.39) yield the properties of the dilatational waves as a function of a frequency variable M and two parameters z_1, z_2 which correspond to the velocities of the P waves without attenuation, as given by equation (5.22). We may rewrite M in terms of f/f_c as:

$$M = \frac{f_c}{f}\frac{\left(\gamma_{12}+\gamma_{22}\right)}{\left(\theta_{11}\theta_{22}-\theta_{12}^2\right)} \tag{5.41}$$

Variables z_I and z_{II} denotes the roots of relation (5.41). Root z_I corresponds to waves of the first kind while z_{II} corresponds to waves of the second kind. Following equations can be obtained:

$$\left(z_I\right)^{\frac{1}{2}} = \mathfrak{R}_I + i\,\mathfrak{I}_I$$
$$\left(z_{II}\right)^{\frac{1}{2}} = \mathfrak{R}_{II} + i\,\mathfrak{I}_{II} \tag{5.42}$$

The phase velocity v_I of the waves of the first kind is given by:

$$\frac{v_I}{V_D} = \frac{1}{\left| \mathfrak{R}_I \right|} \tag{5.43}$$

Attenuation coefficient of the wave of the first kind is given by

$$Im\{k_I\} = \frac{\left| \mathfrak{I}_I \right|}{L_c} \frac{f}{f_c} \tag{5.44}$$

where L_c is a characteristic distance given by:

$$L_c = \frac{V_D}{2\pi f_c} \tag{5.45}$$

The phase velocity and attenuation of the wave of the second kind similarly become:

$$\frac{v_{II}}{V_D} = \frac{1}{\left| \mathfrak{R}_{II} \right|} \tag{5.46}$$

$$Im\{k_{II}\} = \frac{\left| \mathfrak{I}_{II} \right|}{L_c} \frac{f}{f_c} \tag{5.47}$$

Partial summary: We have shown by presenting work of Biot (1956) in this chapter, that there are two P waves and one S wave propagating through poroelastic media in contrast of elastic one, where only one P wave and one S wave are induced. Moreover, we were able to present derivation of velocities for these waves, together with attenuation coefficients.

6 Seismic prospection

In this chapter we will focus our attention on seismic prospection, following and presenting work of de la Puente (2008). Method of seismic prospection uses waves at low- and high-frequencies ranges. The limit between high- and low-frequency ranges is defined by Biot's characteristic frequency:

$$f_B = \min\left(\frac{\eta \phi}{T \kappa \rho_f} \right) \tag{6.1}$$

Equations of motion derived in chapter four are only valid at low frequencies (e.g. seismic frequencies), where fluid flow in pores is laminar (Poiseuille flow) and b is given by (4.4). Physically, at low frequencies $f \le f_B$ Biot's theory states that the wave of second kind becomes extremely dissipative. For homogenous media the wave types propagating in a poroelastic material at low-frequencies are almost indistinguishable from those in a single-phase medium properly attenuated. Low-frequency case has been already discussed in previous chapters five and four, therefore we will only focus on high-frequency range seismic prospection. For high frequencies it is required to introduce viscodynamic operator. This problem will be analyzed in section 6.2, but at first we will introduce anisotropy into constitutive equations (3.17) in section 6.1.

6.1 Anisotropy

At first we must introduce $p = M\left(\varsigma - \alpha\, \varepsilon_{kk}^{m}\right)$ into first equation of (3.17). Now, constitutive equation (3.17) can be extended to general anisotropic case, written in matrix-vector form as

$$\vec{\sigma}_i = N_{ij}\, \vec{\varepsilon}_j \tag{6.2}$$

where

$$\vec{\sigma}_i = \left(\sigma_{xx},\ \ \sigma_{yy},\ \ \sigma_{zz},\ \ \sigma_{yz},\ \ \sigma_{xz},\ \ \sigma_{xy},\ \ -p\right) \tag{6.3}$$

$$\vec{\varepsilon}_j = \left(\varepsilon_{xx},\ \ \varepsilon_{yy},\ \ \varepsilon_{zz},\ \ \varepsilon_{yz},\ \ \varepsilon_{xz},\ \ \varepsilon_{xy},\ \ -\varsigma\right) \tag{6.4}$$

and

$$N_{ij} = \begin{pmatrix} c_{11}^{u} & c_{12}^{u} & c_{13}^{u} & c_{14}^{u} & c_{15}^{u} & c_{16}^{u} & M\alpha_1 \\ c_{12}^{u} & c_{22}^{u} & c_{23}^{u} & c_{24}^{u} & c_{25}^{u} & c_{26}^{u} & M\alpha_2 \\ c_{13}^{u} & c_{23}^{u} & c_{33}^{u} & c_{34}^{u} & c_{35}^{u} & c_{36}^{u} & M\alpha_3 \\ c_{14}^{u} & c_{24}^{u} & c_{34}^{u} & c_{44}^{u} & c_{45}^{u} & c_{46}^{u} & M\alpha_4 \\ c_{15}^{u} & c_{25}^{u} & c_{35}^{u} & c_{45}^{u} & c_{55}^{u} & c_{56}^{u} & M\alpha_5 \\ c_{16}^{u} & c_{26}^{u} & c_{36}^{u} & c_{46}^{u} & c_{56}^{u} & c_{66}^{u} & M\alpha_6 \\ M\alpha_1 & M\alpha_2 & M\alpha_3 & M\alpha_4 & M\alpha_5 & M\alpha_6 & M \end{pmatrix} \tag{6.5}$$

Entries of the matrix (6.5) are $c_{ij}^{u} = c_{ij}^{m} + M\,\alpha_i\,\alpha_j$, which are called components of the undrained stiffness tensor. Undrained stiffness tensor composes of c_{ij}^{m} the components of the elastic Hooke's tensor of the solid matrix, α_i the generalized Biot-Willis's effective stress coefficients and M the Biot's modulus. Parameters α_i and M are denoted as:

$$
\begin{aligned}
\alpha_1 &= 1 - \left(c_{11} + c_{12} + c_{13}\right) / \left(3\, K_S\right) \\
\alpha_2 &= 1 - \left(c_{12} + c_{22} + c_{23}\right) / \left(3\, K_S\right) \\
\alpha_3 &= 1 - \left(c_{13} + c_{23} + c_{33}\right) / \left(3\, K_S\right) \\
\alpha_4 &= \quad - \left(c_{14} + c_{24} + c_{34}\right) / \left(3\, K_S\right) \\
\alpha_5 &= \quad - \left(c_{15} + c_{25} + c_{35}\right) / \left(3\, K_S\right) \\
\alpha_4 &= \quad - \left(c_{16} + c_{26} + c_{36}\right) / \left(3\, K_S\right) \\
M &= \frac{K_S}{\left(1 - \bar{K}/K_S\right) - \phi\left(1 - K_S/K_F\right)} \\
\bar{K} &= \frac{1}{9}\left[c_{11} + c_{22} + c_{33} + 2\left(c_{12} + c_{13} + c_{23}\right)\right]
\end{aligned}
\tag{6.6}
$$

Derivation of relation (6.6) are presented in Appendix A.

6.2 Wave propagation at high frequencies

As we mentioned earlier equation of motion (4.17) won't be valid anymore for high frequencies. It is necessary to introduce viscodynamic operator in term involving b. A general high-frequency viscodynamic operator can be defined in the anisotropic case as

$$
\Psi_i(t) = \frac{\rho_f\, T_i}{\phi}\,\delta(t) + b_i(t) \tag{6.7}
$$

where parameter κ_i is anisotropic permeability, T_i is anisotropic tortuosity of the solid matrix in the principal directions and $b_i(t)$ is dissipation operator. Unfortunately the viscodynamic operator (6.7) is very sensitive to the pore structure and therefore frequency dependence for each material must be analyzed separately. A way around this problem is substituting the convolutional products by a Generalized Maxwell Body. Thus, a phenomenological attenuating law can be used fitted to the experimentally observed wave dispersion for a given material in the high frequency range (de la Puente 2008). Dissipation operator $b_i(t)$ can be expressed by relaxation function $\chi^{(i)}(t)$ for GMB in the following manner

$$b_i(t) = \frac{\eta}{\kappa_i}\chi^{(i)}(t)\,H(t) = \frac{\eta}{\kappa_i}\left[1 - \sum_{l=1}^{n} Y_l^{(i)}\left(1 - e^{-\omega_l t}\right)\right]H(t) \tag{6.8}$$

Equations (4.17) then can be written as:

$$\begin{aligned} \sigma_{ij,j} &= \rho\,\ddot{u}_i + \rho_f\,\ddot{w}_i \\ -p_{,i} &= \rho_f\,\ddot{u}_i + \Psi_i * \ddot{w}_i \end{aligned} \tag{6.9}$$

where $*$ denotes convolutional product in time.

Introducing (6.7) and (6.8) into (6.9), one can obtain:

$$\begin{aligned} \sigma_{ij,j} &= \rho\,\ddot{u}_i + \rho_f\,\ddot{w}_i \\ -p_{,i} &= \rho_f\,\ddot{u}_i + \frac{\rho_f T_i}{\phi}\ddot{w}_i + b_i(t) * \ddot{w}_i \end{aligned} \tag{6.10}$$

We present series of properties for the Dirac's delta function $\delta(t)$ and Heaviside function $H(t)$, which can be summarized as follows:

Property 1: $f(t) * \delta(t) = f(t)$

Property 2: $\dfrac{\partial H(t)}{\partial t} = \delta(t)$

Property 3: $f(t)\,\delta(t) = f(0)\,\delta(t)$

Property 4: $\int_{-\infty}^{\infty} f(a)\,H(t-a) = \int_{-\infty}^{t} f(a)\,da$

Property 5: $f(t) * \left(\dfrac{\partial g(t)}{\partial t}\right) = \left(\dfrac{\partial f(t)}{\partial t}\right) * g(t)$

Using properties 1-5 second equation (6.10) can be rewritten into form:

$$\begin{aligned} -\frac{\partial p}{\partial x_i} &= \rho_f \frac{\partial \dot{u}}{\partial t} + \frac{\rho_f T_i}{\phi}\frac{\partial \dot{w}_i}{\partial t} + \frac{\eta}{\kappa_i}\dot{w}_i - \\ &\frac{\eta}{\kappa_i}\sum_{l=1}^{n} Y_l^{(i)}\,\omega_l \int_{-\infty}^{t} \dot{w}_i(\tau)\,e^{-\omega_l(t-\tau)}\,d\tau \end{aligned} \tag{6.11}$$

It is convenient to introduce a set of anelastic-dynamic variables in vector form $\vec{\vartheta}^{\,l} = \left(\vartheta_x^l,\ \vartheta_y^l,\ \vartheta_z^l\right)^T$ as:

$$\vartheta_i^l = \omega_l \int_{-\infty}^{t} \dot{w}_i(\tau)\,e^{-\omega_l(t-\tau)}\,d\tau \tag{6.12}$$

This leaves the dynamic equations as:

$$-\frac{\partial p}{\partial x_i} = \rho_f \frac{\partial \dot{u}}{\partial t} + \frac{\rho_f T_i}{\phi} \frac{\partial \dot{w}_i}{\partial t} + \frac{\eta}{\kappa_i} \dot{w}_i - \frac{\nu}{\kappa_i} \sum_{l=1}^{n} Y_l^{(i)} \vartheta_i^l \tag{6.13}$$

For anelastic variables we have equation:

$$\frac{\partial}{\partial t} \vartheta_i^l(t) + \omega_l(t) \vartheta_i^l(t) = \omega_l \dot{w}_i(t) \tag{6.14}$$

It should be noted, that the Fourier transform of (6.7) collapses for $\omega \to 0$ into:

$$\Psi_i = \frac{\rho_f T_i}{\phi} \delta(t) + \left(\frac{\eta}{\kappa_i} H(t) \right) \tag{6.15}$$

After adapting this relation to (6.9) and using series of properties 1, 2 and 5 we will obtain:

$$\begin{aligned} \sigma_{ij,j} &= \rho \ \ddot{u}_i + \ \rho_f \ddot{w}_i \\ -p_{,i} &= \rho_f \ddot{u}_i + \frac{T_i}{\phi} \rho_f \ddot{w}_i + \frac{\eta}{\kappa_i} \dot{w}_i \end{aligned} \tag{6.16}$$

If we use isotropic permeability κ and isotropic tortuosity T instead of anisotropic ones, relation (6.16) becomes the same as (4.17). Thus operator (6.7) is consistent also for the low-frequency case. In addition, operator (6.15) is identical to (6.7) for any frequency, in the inviscid case $(\eta = 0)$.

Combining together equations (6.14), (6.11), (6.2) and first equation of (6.9) provides governing equations for wave propagation in porous media as an inhomogeneous linear hyperbolic system of $n_v = 13 + 3n$ first-order partial differential equations that can be expressed in the matrix-vector form

$$\frac{\partial Q_p}{\partial t} + \tilde{A}_{pq} \frac{\partial Q_q}{\partial x} + \tilde{B}_{pq} \frac{\partial Q_q}{\partial y} + \tilde{C}_{pq} \frac{\partial Q_q}{\partial z} = \tilde{E}_{pq} Q_q \tag{6.17}$$

where $p, q \in (1, \cdots, 13)$ denote the elastic part and $p, q \in (14, \cdots, n)$, denote the anelastic part of the system. Note, that classical tensor notation is used in equation (6.17), which implies summation over each index that appears twice. The vector $\vec{Q}$, containing 13 poroelastic variables and $3n$ anelastic-viscodynamic variables, and space-dependent Jacobian matrices $\tilde{A}_{pq}$, $\tilde{B}_{pq}$, $\tilde{C}_{pq}$, $\tilde{E}_{pq}$ are explicitly given as

$$\begin{aligned} \vec{Q} = (&\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{xz}, p, \dot{u}_x, \dot{u}_y, \dot{u}_z, \dot{w}_x, \dot{w}_y, \dot{w}_z, \\ &\vartheta_x^1, \vartheta_y^1, \vartheta_z^1, \dots, \vartheta_x^n, \vartheta_y^n, \vartheta_z^n) \end{aligned} \tag{6.18}$$

$$\tilde{A}_{pq} = \begin{pmatrix} A & 0 \\ A_a & 0 \end{pmatrix}, \quad \tilde{B}_{pq} = \begin{pmatrix} B & 0 \\ B_a & 0 \end{pmatrix}, \quad \tilde{C}_{pq} = \begin{pmatrix} C & 0 \\ C_a & 0 \end{pmatrix} \tag{6.19}$$

where $\tilde{A}_{pq},\ \tilde{B}_{pq},\ \tilde{C}_{pq} \in \mathbb{R}^{n_v \times n_v}$ containing matrices $A,\ B,\ C \in \mathbb{R}^{13\times 13}$ responsible for poroelastic part, and $A_a,\ B_a,\ C_a \in \mathbb{R}^{3n\times 13}$ responsible for anelastic part. Matrices $A,\ B,\ C,\ A_a,\ B_a,\ C_a$ have form

$$A = -\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & c_{11}^u & c_{16}^u & c_{15}^u & 0 & M\alpha_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{12}^u & c_{26}^u & c_{25}^u & 0 & M\alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{13}^u & c_{36}^u & c_{35}^u & 0 & M\alpha_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{16}^u & c_{66}^u & c_{56}^u & 0 & M\alpha_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{14}^u & c_{46}^u & c_{45}^u & 0 & M\alpha_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{15}^u & c_{56}^u & c_{55}^u & 0 & M\alpha_5 & 0 & 0 \\
\frac{1}{\rho_x^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_x^{(1)}}{\rho_x^{(1)}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\rho_y^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_z^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -M\alpha_1 & -M\alpha_6 & -M\alpha_5 & 0 & -M & 0 & 0 \\
\frac{1}{\rho_x^{(2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta^{(2)}}{\rho_x^{(2)}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\rho_y^{(2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_z^{(2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{6.20}$$

$$B = -\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & c_{16}^{u} & c_{12}^{u} & c_{14}^{u} & 0 & 0 & M\,\alpha_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{26}^{u} & c_{22}^{u} & c_{24}^{u} & 0 & 0 & M\,\alpha_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{36}^{u} & c_{23}^{u} & c_{34}^{u} & 0 & 0 & M\,\alpha_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{66}^{u} & c_{26}^{u} & c_{46}^{u} & 0 & 0 & M\,\alpha_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{46}^{u} & c_{24}^{u} & c_{44}^{u} & 0 & 0 & M\,\alpha_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{56}^{u} & c_{25}^{u} & c_{45}^{u} & 0 & 0 & M\,\alpha_5 & 0 \\
0 & 0 & 0 & \frac{1}{\rho_x^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\rho_y^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_y^{(1)}}{\rho_y^{(1)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\rho_z^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -M\,\alpha_6 & -M\,\alpha_2 & -M\,\alpha_4 & 0 & 0 & -M & 0 \\
0 & 0 & 0 & \frac{1}{\rho_x^{(2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\rho_y^{(2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta^{(2)}}{\rho_y^{(2)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\rho_z^{(2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{6.21}$$

$$C = -\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & c_{15}^{u} & c_{14}^{u} & c_{13}^{u} & 0 & 0 & 0 & M\,\alpha_1 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{25}^{u} & c_{24}^{u} & c_{23}^{u} & 0 & 0 & 0 & M\,\alpha_2 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{35}^{u} & c_{34}^{u} & c_{33}^{u} & 0 & 0 & 0 & M\,\alpha_3 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{56}^{u} & c_{46}^{u} & c_{36}^{u} & 0 & 0 & 0 & M\,\alpha_6 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{45}^{u} & c_{44}^{u} & c_{34}^{u} & 0 & 0 & 0 & M\,\alpha_4 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{55}^{u} & c_{45}^{u} & c_{35}^{u} & 0 & 0 & 0 & M\,\alpha_5 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_x^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\rho_y^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\rho_z^{(1)}} & 0 & & 0 & 0 & 0 & 0 & \frac{\beta_z^{(1)}}{\rho_z^{(1)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -M\,\alpha_5 & -M\,\alpha_4 & -M\,\alpha_3 & 0 & 0 & 0 & -M \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_x^{(2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\rho_y^{(2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\rho_z^{(2)}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta^{(2)}}{\rho_y^{(2)}} & 0 & 0 & 0
\end{pmatrix} \tag{6.22}$$

$$A_a = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}, \quad B_a = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix}, \quad C_a = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} \tag{6.23}$$

where entries $\rho_i^{(1)}, \rho_i^{(2)}, \beta_i^{(1)}, \beta_i^{(2)}$ of matrices $A,\ B,\ C$ has form:

$$\begin{aligned} \rho_i^{(1)} &= \rho - \phi\,\rho_f / T_i, \qquad \beta_i^{(1)} = \phi / T_i \\ \rho_i^{(2)} &= \rho_f - T_i\,\rho / \phi, \qquad \beta^{(2)} = \rho / \rho_f \end{aligned} \tag{6.24}$$

Matrices $A_a,\ B_a,\ C_a$ contain sub-matrices $A_l,\ B_l,\ C_l \in \mathbb{R}^{3\times 13}$, with $l = 1,\dots,n$, in the form

$$A_l = \omega_l \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.25}$$

$$B_l = \omega_l \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.26}$$

$$C_l = \omega_l \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.27}$$

where ω_l is the relaxation frequency of the l-th mechanism.

The reaction source in (6.17), which couples the anelastic functions to the original elastic system can be represented by matrix $\tilde{E}$ in the form

$$\tilde{E} = \begin{pmatrix} E & E' \\ E'' & E''' \end{pmatrix} \in \mathbb{R}^{n_v \times n_v} \tag{6.28}$$

where $E \in \mathbb{R}^{13\times 13}$ has a structure:

$$E = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_x^{(1)}\,\nu}{\rho_x^{(1)}\,\kappa_x} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_y^{(1)}\,\nu}{\rho_y^{(1)}\,\kappa_y} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_z^{(1)}\,\nu}{\rho_z^{(1)}\,\kappa_z} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_x^{(2)}\,\nu}{\rho_x^{(2)}\,\kappa_x} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_x^{(2)}\,\nu}{\rho_y^{(2)}\,\kappa_y} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_x^{(2)}\,\nu}{\rho_z^{(2)}\,\kappa_z}
\end{pmatrix} \tag{6.29}$$

The matrix E' from relation (6.28) has the block structure

$$E' = \left(E'_1, \ldots, E'_n \right) \in \mathbb{R}^{13 \times 3n} \tag{6.30}$$

where each matrix $E'_l \in \mathbb{R}^{13 \times 3}$, with $l = 1, \ldots, n$, contains the elastic-dynamic coefficients $Y_l^{(i)}$ of the l-th mechanism in the form

$$E_l' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\beta_x^{(1)} \nu}{\rho_x^{(1)} \kappa_x} Y_l^{(x)} & 0 & 0 \\ 0 & -\frac{\beta_y^{(1)} \nu}{\rho_y^{(1)} \kappa_y} Y_l^{(y)} & 0 \\ 0 & 0 & -\frac{\beta_z^{(1)} \nu}{\rho_z^{(1)} \kappa_z} Y_l^{(z)} \\ 0 & 0 & 0 \\ -\frac{\beta^{(2)} \nu}{\rho_x^{(2)} \kappa_x} Y_l^{(x)} & 0 & 0 \\ 0 & -\frac{\beta^{(2)} \nu}{\rho_x^{(2)} \kappa_x} Y_l^{(x)} & 0 \\ 0 & 0 & -\frac{\beta^{(2)} \nu}{\rho_x^{(2)} \kappa_x} Y_l^{(x)} \end{pmatrix} \tag{6.31}$$

The matrix E''' in (6.28) is a diagonal matrix and has a structure

$$E''' = \begin{pmatrix} E_1''' & & 0 \\ & \ddots & \\ 0 & & E_n''' \end{pmatrix} \in \mathbb{R}^{3n \times 3n} \tag{6.32}$$

where each matrix $E_l''' \in \mathbb{R}^{3 \times 3}$, with $l = 1, \ldots, n$, is itself a diagonal matrix containing only the relaxation frequency ω_l of the l-th mechanism on its diagonal, i.e. $E_l''' = -\omega_l \cdot I$ with $I \in \mathbb{R}^{3 \times 3}$ denoting the identity matrix.

Finally the E'' block in (6.28) has a form

$$E'' = \begin{pmatrix} E_1'' \\ \vdots \\ E_n'' \end{pmatrix} \in \mathbb{R}^{3n \times 13} \tag{6.33}$$

where each sub-matrix $E_l'' \in \mathbb{R}^{3 \times 13}$, with $l = 1, \ldots, n$, contains the relaxation frequency ω_l of the l-th mechanism in the form

$$E_l'' = \omega_l \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.34}$$

7 Attenuation of seismic waves

In this chapter we will present derivation of equations of motion, which contain expression responsible for attenuation of seismic waves due to anelasticity of the frame. It should be noted, that these dissipation and attenuation processes are not caused by viscous resistance to fluid flow. Here is introduced only dissipation phenomena of mechanical, chemical or thermomechanical nature, associated with the anelasticity of the frame, which are usually taken into account by introducing a viscoelastic rheology.

From chapter three and four we know, that constitutive equations and equations of motions have following structure:

$$\begin{aligned} \sigma_{ij} &= \lambda\, \delta_{ij}\, \varepsilon_{kk}^m + 2\, G\, \varepsilon_{ij}^m - \delta_{ij}\, \alpha\, M\, \varsigma \\ p &= -\alpha\, M\, \varepsilon_{kk}^m + \qquad\qquad M\, \varsigma \end{aligned} \tag{7.1}$$

$$\begin{aligned} \sigma_{ij,j} &= \rho\ \ddot{u}_i + \quad \rho_f\, \ddot{w}_i \\ -p_{,i} &= \rho_f\, \ddot{u}_i + \frac{T}{\phi}\rho_f\, \ddot{w}_i + b\, \dot{w}_i \end{aligned} \tag{7.2}$$

As we mentioned earlier relation (7.2) is only valid at low frequencies. Our interest is to study wave propagation in porous material at seismic frequency, which belongs to low-frequency range according to O'Brien (2010) and Masson (2007). Unfortunately, constitutive equations (7.1) do not contain attenuation controlled by the anelasticity of the frame, therefore we have to introduce viscoelastic rheology.

New form of relation (7.1) may be obtained by introducing some of the parameters mentioned in chapter three and four:

$$\begin{aligned} \sigma_{ij} &= K_M\, \delta_{ij}\, \varepsilon_{kk}^m + 2\, G\left(\varepsilon_{ij}^m - \frac{1}{3}\varepsilon_{kk}^m\, \delta_{ij}\right) + \alpha^2\, M\, \delta_{ij}\, \varepsilon_{kk}^m + \alpha\, M\, \partial_i\, w_i \\ \sigma^f &= \qquad \alpha\, M \varepsilon_{kk}^m + \qquad\qquad\qquad\qquad\qquad M\, \partial_i\, w_i \end{aligned} \tag{7.3}$$

where

$$M = \frac{K_S}{1 - \phi - K_M / K_S + \phi\, K_S / K_F}, \qquad \alpha = 1 - \frac{K_M}{K_S} \tag{7.4}$$

At this point we can make same assumption as Morency (2008), that only the time-dependence of the bulk and shear moduli of the frame, K_M and G, needs to be considered, accommodating the fact that various forms of energy dissipation may occur at grain contacts.

In practice, attenuation in the Earth is mainly controlled by the shear quality factor, such that only the time dependence of the isotropic shear modulus G need be accommodated.

For modelling a variety of dissipation mechanisms related to the skeleton-fluid interaction we will use rheological model of General Maxwell body, which based upon using linear combinations in parallel of so-called Maxwell Bodies, essentially a spring and a dashpot connected in series.

Then we can express constitutive equations (7.1) as:

$$\begin{aligned} \sigma_{ij} &= K_M \, \delta_{ij} \, \varepsilon_{kk}^m + \alpha M \, \partial_i w_i + \alpha^2 M \, \delta_{ij} \, \varepsilon_{kk}^m + \\ & 2G\left(\varepsilon_{ij}^m - \frac{1}{3} \varepsilon_{kk}^m \, \delta_{ij} \right) - \sum_{l=1}^{n} 2 Y_l^G \, G \left({}^m\zeta_l^{ij} - \frac{1}{3} {}^m\zeta_l^{kk} \, \delta_{ij} \right) \\ \sigma^f &= \alpha M \, \varepsilon_{kk}^m + M \, \partial_i w_i \end{aligned} \tag{7.5}$$

The anelastic functions (Moczo 2014) are solutions of the differential equations

$$\frac{\partial}{\partial t} \zeta_l^{ij}(t) + \omega_l \, \zeta_l^{ij}(t) = \omega_l \, \varepsilon_{ij}(t), \quad l = 1, \ldots, n \tag{7.6}$$

The equal-index summation convention applies to spatial index k but does not apply to subscript l. For n characteristic frequencies ω_l we have n anelastic coefficients Y_l^G.

Make partial summary: Equations of motion (7.1) and constitutive equations (7.5), together with differential equations (7.6) for anelastic functions generate system of equation describing wave propagation of seismic waves in poroelastic medium with anelastic frame.

8 Algorithmical preparation

This last chapter is devoted to algorithm preparation of theory of poroelasticity for modelling of seismic propagation using finite-difference method in 3D. We are going to express our constitutive equations and equations of motion in velocity-stress formulation. Numerical scheme uses uniform staggered space-time grid and it is a 2^{nd}-order accuracy in time and 4^{nd}-order accuracy in space.

The explicit expression of the constitutive equations (3.18) and poroelastic wave equations (4.16) can be written in velocity-stress formulation in 3D case as

$$\frac{\partial \sigma_{xx}}{\partial t} = 2G\left(\frac{\partial V_x}{\partial x}\right) + \lambda_M\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right) +$$

$$\alpha M\left(\frac{\partial W_x}{\partial x}+\frac{\partial W_y}{\partial y}+\frac{\partial W_z}{\partial z}\right) + \alpha^2 M\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)$$

$$\frac{\partial \sigma_{yy}}{\partial t} = 2G\left(\frac{\partial V_y}{\partial y}\right) + \lambda_M\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right) +$$

$$\alpha M\left(\frac{\partial W_x}{\partial x}+\frac{\partial W_y}{\partial y}+\frac{\partial W_z}{\partial z}\right) + \alpha^2 M\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)$$

$$\frac{\partial \sigma_{zz}}{\partial t} = 2G\left(\frac{\partial V_z}{\partial z}\right) + \lambda_M\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right) +$$

$$\alpha M\left(\frac{\partial W_x}{\partial x}+\frac{\partial W_y}{\partial y}+\frac{\partial W_z}{\partial z}\right) + \alpha^2 M\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)$$

$$\frac{\partial \sigma_{xy}}{\partial t} = G\left(\frac{\partial V_x}{\partial y}+\frac{\partial V_y}{\partial x}\right)$$

$$\frac{\partial \sigma_{yz}}{\partial t} = G\left(\frac{\partial V_y}{\partial z}+\frac{\partial V_z}{\partial y}\right)$$

$$\frac{\partial \sigma_{xz}}{\partial t} = G\left(\frac{\partial V_x}{\partial z}+\frac{\partial V_z}{\partial x}\right)$$

$$-\frac{\partial p}{\partial t} = M\left(\frac{\partial W_x}{\partial x}+\frac{\partial W_y}{\partial y}+\frac{\partial W_z}{\partial z}\right) + \alpha M\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right) \tag{8.1}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \frac{\partial V_x}{\partial t} + \rho_f \frac{\partial W_x}{\partial t}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \rho \frac{\partial V_y}{\partial t} + \rho_f \frac{\partial W_y}{\partial t}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial V_z}{\partial t} + \rho_f \frac{\partial W_z}{\partial t}$$

$$-\frac{\partial p}{\partial x} = \rho_f \frac{\partial V_x}{\partial t} + \frac{T}{\phi} \rho_f \frac{\partial W_x}{\partial t} + \frac{\eta}{\kappa} W_x$$

$$-\frac{\partial p}{\partial y} = \rho_f \frac{\partial V_y}{\partial t} + \frac{T}{\phi} \rho_f \frac{\partial W_y}{\partial t} + \frac{\eta}{\kappa} W_y$$

$$-\frac{\partial p}{\partial z} = \rho_f \frac{\partial V_z}{\partial t} + \frac{T}{\phi} \rho_f \frac{\partial W_z}{\partial t} + \frac{\eta}{\kappa} W_z \qquad (8.2)$$

where V_x, V_y, V_z and W_x, W_y, W_z are solid particle velocities and relative solid to fluid velocities.

In order to discretize the equations of motion (8.2) on staggered grids for finite-difference algorithms, we rewrite the equations into the following form

$$\frac{\partial V_x}{\partial t} = \frac{1}{\frac{T}{\phi}\rho - \rho_f}\left[\frac{T}{\phi}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right) + \frac{\partial p}{\partial x} + \frac{\eta}{\kappa}W_x\right]$$

$$\frac{\partial V_y}{\partial t} = \frac{1}{\frac{T}{\phi}\rho - \rho_f}\left[\frac{T}{\phi}\left(\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y}\right) + \frac{\partial p}{\partial y} + \frac{\eta}{\kappa}W_y\right]$$

$$\frac{\partial V_z}{\partial t} = \frac{1}{\frac{T}{\phi}\rho - \rho_f}\left[\frac{T}{\phi}\left(\frac{\partial \sigma_{zz}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y}\right) + \frac{\partial p}{\partial z} + \frac{\eta}{\kappa}W_z\right]$$

$$\frac{\partial W_x}{\partial t} = \frac{1}{\frac{\rho_f^2}{\rho} - \frac{T\rho_f}{\phi}}\left[\frac{\rho_f}{\rho}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right) + \frac{\partial p}{\partial x} + \frac{\eta}{\kappa}W_x\right]$$

$$\frac{\partial W_y}{\partial t} = \frac{1}{\frac{\rho_f^2}{\rho} - \frac{T\rho_f}{\phi}}\left[\frac{\rho_f}{\rho}\left(\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y}\right) + \frac{\partial p}{\partial y} + \frac{\eta}{\kappa}W_y\right] \qquad (8.3)$$

$$\frac{\partial W_z}{\partial t} = \frac{1}{\frac{\rho_f^2}{\rho} - \frac{T\rho_f}{\phi}}\left[\frac{\rho_f}{\rho}\left(\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y}\right) + \frac{\partial p}{\partial z} + \frac{\eta}{\kappa}W_z\right]$$

Denote the discrete grid values of the particle velocity components $V_x, V_y, V_z, W_x, W_y, W_z$ by VX, VY, VZ, WX, WY, WZ . Similarly denote the stress-tensor components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ by $TXX, TYY, TZZ, TXY, TXZ, TYZ$ and pore pressure p by P . Figure 8 shows the staggered grid cell of (2,4) velocity-stress scheme. We may approximate the first of equation (8.3) at the time level m and spatial position $I, K+1/2, L+1/2$

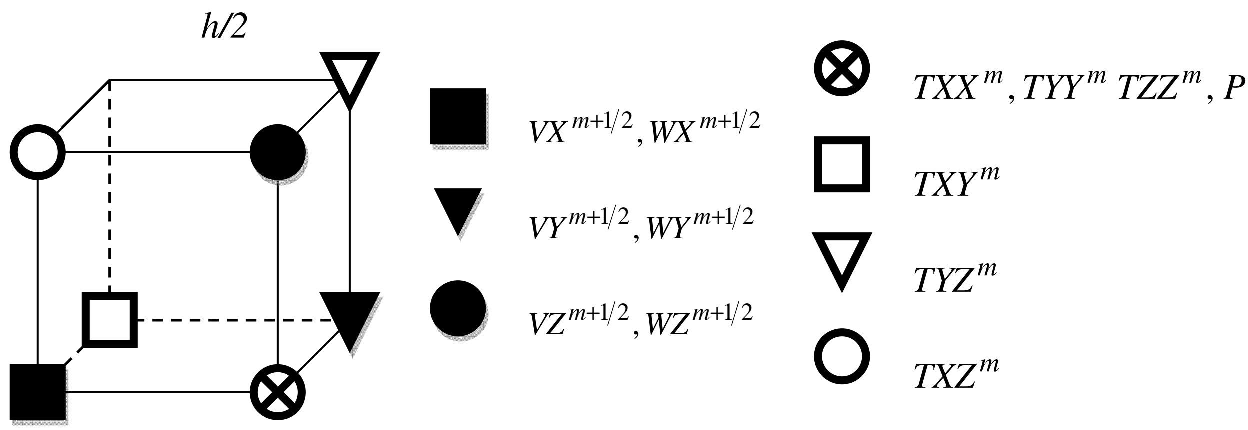

Figure 8 Grid cell in the staggered grid.

It should be noted, that we assume smoothly and weakly heterogeneous isotropic poroelastic unbounded medium. For brevity we will only consider equations for V_x, W_x and σ_{xx}, p. The staggered format of equations (8.1) and (8.3) can be written as

$$TXX^{m}_{I+1/2,\,K+1/2,\,L+1/2} = TXX^{m-1}_{I+1/2,\,K+1/2,\,L+1/2}$$

$$+\ \frac{\Delta}{h}\Bigg\{\left(\lambda_M + 2\,G\right)_{I+1/2,\,K+1/2,,\,L+1/2}\left[\frac{9}{8}\left(VX^{m-1/2}_{I+1,\,K+1/2,\,L+1/2} - VX^{m-1/2}_{I,\,K+1/2,\,L+1/2}\right)\right.$$

$$\left.-\frac{1}{24}\left(VX^{m-1/2}_{I+2,\,K+1/2,\,L+1/2} - VX^{m-1/2}_{I-1,\,K+1/2,\,L+1/2}\right)\right]$$

$$+\left(\lambda_M\right)_{I+1/2,\,K+1/2,\,L+1/2}\left[\frac{9}{8}\left(VY^{m-1/2}_{I+1/2,\,K+1,\,L+1/2} - VY^{m-1/2}_{I+1/2,\,K,\,L+1/2}\right)\right.$$

$$\left.-\frac{1}{24}\left(VY^{m-1/2}_{I+1/2,\,K+2,\,L+1/2} - VY^{m-1/2}_{I+1/2,\,K-1,\,L+1/2}\right)\right]$$

$$+\left(\lambda_M\right)_{I+1/2,\,K+1/2,\,L+1/2}\left[\frac{9}{8}\left(VZ^{m-1/2}_{I+1/2,\,K+1/2,\,L+1} - VZ^{m-1/2}_{I+1/2,\,K+1/2,\,L}\right)\right.$$

$$\left.-\frac{1}{24}\left(VY^{m-1/2}_{I+1/2,\,K+1/2,\,L+2} - VY^{m-1/2}_{I+1/2,\,K+1/2,\,L-1}\right)\right]$$

$$+\left(\alpha\,M\right)_{I+1/2,\,K+1/2,\,L+1/2}\left[\frac{9}{8}\left(WX^{m-1/2}_{I+1,\,K+1/2,\,L+1/2} - WX^{m-1/2}_{I,\,K+1/2,\,L+1/2}\right)\right.$$

$$\left.-\frac{1}{24}\left(WX^{m-1/2}_{I+2,\,K+1/2,\,L+1/2} - WX^{m-1/2}_{I-1,\,K+1/2,\,L+1/2}\right)\right]$$

$$+\left(\alpha\,M\right)_{I+1/2,\,K+1/2,\,L+1/2}\left[\frac{9}{8}\left(WY^{m-1/2}_{I+1/2,\,K+1,\,L+1/2} - WY^{m-1/2}_{I+1/2,\,K,\,L+1/2}\right)\right.$$

$$\left.-\frac{1}{24}\left(WY^{m-1/2}_{I+1/2,\,K+2,\,L+1/2} - WY^{m-1/2}_{I+1/2,\,K-1,\,L+1/2}\right)\right] \qquad (8.4)$$

$$+(\alpha M)_{I+1/2,\,K+1/2,\,L+1/2}\left[\frac{9}{8}\left(WZ^{m-1/2}_{I+1/2,\,K+1/2,\,L+1}-WZ^{m-1/2}_{I+1/2,\,K+1/2,\,L}\right)\right.$$

$$\left.-\frac{1}{24}\left(WZ^{m-1/2}_{I+1/2,\,K+1/2,\,L+2}-WZ^{m-1/2}_{I+1/2,\,K+1/2,\,L-1}\right)\right]$$

$$+\left(\alpha^2 M\right)_{I+1/2,\,K+1/2,\,L+1/2}\left[\frac{9}{8}\left(VX^{m-1/2}_{I+1,\,K+1/2,\,L+1/2}-VX^{m-1/2}_{I,\,K+1/2,\,L+1/2}\right)\right.$$

$$\left.-\frac{1}{24}\left(VX^{m-1/2}_{I+2,\,K+1/2,\,L+1/2}-VX^{m-1/2}_{I-1,\,K+1/2,\,L+1/2}\right)\right]$$

$$+\left(\alpha^2 M\right)_{I+1/2,\,K+1/2,\,L+1/2}\left[\frac{9}{8}\left(VY^{m-1/2}_{I+1/2,\,K+1,\,L+1/2}-VY^{m-1/2}_{I+1/2,\,K,\,L+1/2}\right)\right.$$

$$\left.-\frac{1}{24}\left(VY^{m-1/2}_{I+1/2,\,K+2,\,L+1/2}-VY^{m-1/2}_{I+1/2,\,K-1,\,L+1/2}\right)\right]$$

$$+\left(\alpha^2 M\right)_{I+1/2,\,K+1/2,\,L+1/2}\left[\frac{9}{8}\left(VZ^{m-1/2}_{I+1/2,\,K+1/2,\,L+1}-VY^{m-1/2}_{I+1/2,\,K+1/2,\,L}\right)\right.$$

$$\left.\left.-\frac{1}{24}\left(VZ^{m-1/2}_{I+1/2,\,K+1/2,\,L+2}-VZ^{m-1/2}_{I+1/2,\,K+1/2,\,L-1}\right)\right]\right\}$$

$$
\begin{aligned}
-P^{m}_{I+1/2,\,K+1/2,\,L+1/2} &= -P^{m-1}_{I+1/2,\,K+1/2,\,L+1/2} \\
&+\frac{\Delta}{h}\Bigg\{M_{I+1/2,\,K+1/2,\,L+1/2}\bigg[\frac{9}{8}\Big(WX^{m-1/2}_{I+1,\,K+1/2,\,L+1/2}-WX^{m-1/2}_{I,\,K+1/2,\,L+1/2}\Big) \\
&\qquad -\frac{1}{24}\Big(WX^{m-1/2}_{I+2,\,K+1/2,\,L+1/2}-WX^{m-1/2}_{I-1,\,K+1/2,\,L+1/2}\Big) \\
&\qquad +\frac{9}{8}\Big(WY^{m-1/2}_{I+1/2,\,K+1,\,L+1/2}-WY^{m-1/2}_{I+1/2,\,K,\,L+1/2}\Big) \\
&\qquad -\frac{1}{24}\Big(WY^{m-1/2}_{I+1/2,\,K+2,\,L+1/2}-WY^{m-1/2}_{I+1/2,\,K-1,\,L+1/2}\Big) \\
&\qquad +\frac{9}{8}\Big(WZ^{m-1/2}_{I+1/2,\,K+1/2,\,L+1}-WZ^{m-1/2}_{I+1/2,\,K+1/2,\,L}\Big) \\
&\qquad -\frac{1}{24}\Big(WZ^{m-1/2}_{I+1/2,\,K+2,\,L+2}-WZ^{m-1/2}_{I+1/2,\,K-1,\,L-1}\Big)\bigg] \\
&+(\alpha M)_{I+1/2,\,K+1/2,\,L+1/2}\bigg[\frac{9}{8}\Big(VX^{m-1/2}_{I+1,\,K+1/2,\,L+1/2}-VX^{m-1/2}_{I,\,K+1/2,\,L+1/2}\Big) \\
&\qquad -\frac{1}{24}\Big(VX^{m-1/2}_{I+2,\,K+1/2,\,L+1/2}-VX^{m-1/2}_{I-1,\,K+1/2,\,L+1/2}\Big) \\
&\qquad +\frac{9}{8}\Big(VY^{m-1/2}_{I+1/2,\,K+1,\,L+1/2}-VY^{m-1/2}_{I+1/2,\,K,\,L+1/2}\Big) \\
&\qquad -\frac{1}{24}\Big(VY^{m-1/2}_{I+1/2,\,K+2,\,L+1/2}-VY^{m-1/2}_{I+1/2,\,K-1,\,L+1/2}\Big) \\
&\qquad +\frac{9}{8}\Big(VZ^{m-1/2}_{I+1/2,\,K+1/2,\,L+1}-VZ^{m-1/2}_{I+1/2,\,K+1/2,\,L}\Big) \\
&\qquad -\frac{1}{24}\Big(VZ^{m-1/2}_{I+1/2,\,K+2,\,L+2}-VZ^{m-1/2}_{I+1/2,\,K-1,\,L-1}\Big)\bigg]\Bigg\}
\end{aligned}
\tag{8.5}
$$

and

$$VX^{m+1/2}_{I,\,K+1/2,\,L+1/2} = VX^{m-1/2}_{I,\,K+1/2,\,L+1/2}$$

$$+ \frac{\Delta}{h}\frac{1}{\frac{T\,\rho_{I,\,K+1/2,\,L+1/2}}{\phi} - \left(\rho_f\right)_{I,\,K+1/2,\,L+1/2}}\left\{\left[\frac{T}{\phi}\frac{9}{8}\left(TXX^m_{I+1/2,\,K+1/2,\,L+1/2} - TXX^m_{I-1/2,\,K+1/2,\,L+1/2}\right)\right.\right.$$

$$-\frac{1}{24}\left(TXX^m_{I+3/2,\,K+1/2,\,L+1/2} - TXX^m_{I-3/2,\,K+1/2,\,L+1/2}\right)$$

$$+\frac{9}{8}\left(TXY^m_{I,\,K+1,\,L+1/2} - TXY^m_{I,\,K,\,L+1/2}\right)$$

$$-\frac{1}{24}\left(TXY^m_{I,\,K+2,\,L+1/2} - TXY^m_{I,\,K-1,\,L+1/2}\right)$$

$$+\frac{9}{8}\left(TXZ^m_{I,\,K+1/2,\,L+1} - TXZ^m_{I,\,K+1/2,\,L}\right)$$

$$\left.-\frac{1}{24}\left(TXZ^m_{I,\,K+1/2,\,L+2} - TXZ^m_{I,\,K+1/2,\,L-1}\right)\right]$$

$$+\left[\frac{9}{8}\left(P^m_{I+1/2,\,K+1/2,\,L+1/2} - P^m_{I-1/2,\,K+1/2,\,L+1/2}\right)\right.$$

$$\left.-\frac{1}{24}\left(P^m_{I+3/2,\,K+1/2,\,L+1/2} - P^m_{I-3/2,\,K+1/2,\,L+1/2}\right)\right]$$

$$\left.+\frac{\eta}{\kappa}\left(WX^m_{I,\,K+1/2,\,L+1/2}\right)\right\}$$

(8.6)

$$WX^{m+1/2}_{I,\,K+1/2,\,L+1/2} = WX^{m-1/2}_{I,\,K+1/2,\,L+1/2}$$

$$+\frac{\Delta}{h}\frac{1}{\frac{\left(\rho_f^2\right)_{I,\,K+1/2,\,L+1/2}}{\rho}-\frac{T\left(\rho_f\right)_{I,\,K+1/2,\,L+1/2}}{\phi}}\left\{\left(\frac{\rho_f}{\rho}\right)_{I,\,K+1/2,\,L+1/2}\frac{9}{8}\left(TXX^m_{I+1/2,\,K+1/2,\,L+1/2}-TXX^m_{I-1/2,\,K+1/2,\,L+1/2}\right)\right.$$

$$-\frac{1}{24}\left(TXX^m_{I+3/2,\,K+1/2,\,L+1/2}-TXX^m_{I-3/2,\,K+1/2,\,L+1/2}\right)$$

$$+\frac{9}{8}\left(TXY^m_{I,\,K+1,\,L+1/2}-TXY^m_{I,\,K,\,L+1/2}\right)$$

$$-\frac{1}{24}\left(TXY^m_{I,\,K+2,\,L+1/2}-TXY^m_{I,\,K-1,\,L+1/2}\right)\Big]$$

$$+\frac{9}{8}\left(TXZ^m_{I,\,K+1/2,\,L+1}-TXZ^m_{I,\,K+1/2,\,L}\right)$$

$$-\frac{1}{24}\left(TXZ^m_{I,\,K+1/2,\,L+2}-TXZ^m_{I,\,K+1/2,\,L-1}\right)\Big]$$

$$+\Big[\frac{9}{8}\left(P^m_{I+1/2,\,K+1/2,\,L+1/2}-P^m_{I-1/2,\,K+1/2,\,L+1/2}\right)$$

$$-\frac{1}{24}\left(P^m_{I+3/2,\,K+1/2,\,L+1/2}-P^m_{I-3/2,\,K+1/2,\,L+1/2}\right)\Big]$$

$$\left.+\frac{\eta}{\kappa}\left(WX^m_{I,\,K+1/2,\,L+1/2}\right)\right\}$$

(8.7)

where $\left(\lambda_M\right)_{I+1/2,\,K+1/2,\,L+1/2}$, $G_{I+1/2,\,K+1/2,\,L+1/2}$, $\alpha_{I+1/2,\,K+1/2,\,L+1/2}$, $M_{I+1/2,\,K+1/2,\,L+1/2}$, $\left(\rho_f\right)_{I,\,K+1/2,\,L+1/2}$, $\rho_{I,\,K+1/2,\,L+1/2}$ are effective grid material parameters, defined as an integral harmonic averages:

$$\left(\lambda_M\right)_{I+1/2,\, K+1/2, L+1/2} = \left[\frac{1}{h^2}\int_{x_I}^{x_{I+1}}\int_{y_K}^{y_{K+1}}\int_{z_L}^{z_{L+1}}\frac{1}{\lambda_M}\,dx\,dy\,dz\right]^{-1}$$

$$G_{I+1/2,\, K+1/2, L+1/2} = \left[\frac{1}{h^2}\int_{x_I}^{x_{I+1}}\int_{y_K}^{y_{K+1}}\int_{z_L}^{z_{L+1}}\frac{1}{G}\,dx\,dy\,dz\right]^{-1}$$

$$\alpha_{I+1/2,\, K+1/2, L+1/2} = \left[\frac{1}{h^2}\int_{x_I}^{x_{I+1}}\int_{y_K}^{y_{K+1}}\int_{z_L}^{z_{L+1}}\frac{1}{\alpha}\,dx\,dy\,dz\right]^{-1}$$

$$M_{I+1/2,\, K+1/2, L+1/2} = \left[\frac{1}{h^2}\int_{x_I}^{x_{I+1}}\int_{y_K}^{y_{K+1}}\int_{z_L}^{z_{L+1}}\frac{1}{M}\,dx\,dy\,dz\right]^{-1}$$

$$\left(\rho_f\right)_{I,\, K+1/2,\, L+1/2} = \left[\frac{1}{h^2}\int_{x_{I-1/2}}^{x_{I+1/2}}\int_{y_K}^{y_{K+1}}\int_{z_L}^{z_{L+1}}\frac{1}{\rho_f}\,dx\,dy\,dz\right]^{-1}$$

$$\rho_{I,\, K+1/2,\, L+1/2} = \left[\frac{1}{h^2}\int_{x_{I-1/2}}^{x_{I+1/2}}\int_{y_K}^{y_{K+1}}\int_{z_L}^{z_{L+1}}\frac{1}{\rho}\,dx\,dy\,dz\right]^{-1} \qquad (8.8)$$

Another parameters, such as ϕ, T, η, κ are assumed to be homogenous.

Relations (8.4), (8.5), (8.6), (8.7), (8.8), together with equations for $TYY, TZZ, TXY,, TXZ, TYZ, VY, VZ, WY, WZ$ form velocity-stress scheme on staggered grid, that will be used for numerical simulation of seismic motion in our master thesis.

Conclusions

In this thesis we presented:

- an introductory text on the theory of poroelasticity, starting by defining the poroelastic medium as a material containing pores that are typically filled with a fluid,
- basic terms, parameters, assumptions and conditions, which are essential for derivation of constitutive equations and equations of motion,
- derivation of the constitutive equations for isotropic and anisotropic poroelastic media,
- derivation of the equations of motion for a poroelastic material at a low-frequency range on the base of Hamilton's principle,
- demonstration of the existence of wave of second kind, also denoted as slow P wave,
- behavior of one S wave and two P waves, together with derivation of velocities and attenuation coefficients for these waves,

- equations of motion in poroelastic media at high-frequency range using the General Maxwell body and viscodynamic operator,
- implementation of mechanical attenuation into equations of motion at low frequencies, caused by anelasticity of the frame,
- staggered-grid finite-difference scheme for numerical modelling of seismic motion.

The presented material can serve a sufficient theoretical basis for possible future elaboration in the team of supervisor:

- modification of the constitutive relations and equations of motion for the case of thermoporoelasticity to describe dynamic fault weakening mechanism, known as thermal pressurization of pore fluid,
- numerical simulation of seismic wave propagation and earthquake motion for a poroviscoelastic model which makes it possible to account for attenuation due to anelasticity of the real Earth 's material,
- combine Iwan model with poroelastic rheology.

Appendix A

In this appendix we focus on the presentation of derivation of relations in equation (6.6). Let's begin with generalized Biot-Willis's effective stress coefficients α_{ij}.

In chapter four we have introduced effective stress-strain relation denoted as:

$$\sigma_{ij}^{\mathrm{eff}} = c_{ijkl}^{m} \, \varepsilon_{kl}^{m} \tag{A.1}$$

It should be noted, that equation (A.1) can be inverted to strain-stress equation related to s_{ijkl}^{m}, where s_{ijkl}^{m} is the compliance tensor, which satisfy this condition:

$$c_{ijkl} s_{klrs} = \frac{1}{2}\left(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}\right) \tag{A.2}$$

Now we can continue deriving formula for stress coefficient for anisotropic case. This proof is based on work of Cowin (2013) and Carcione (2001).

Consider now a representative elementary volume of saturated porous medium. It is bounded by the outer surface S_0 and by inner surface S_{p} (pore boundaries). Let us consider the stress vector:

$$\begin{aligned} t_i^{0} &= \sigma_{ij} \, n_j \quad \text{on} \quad S_0 \\ t_i^{\mathrm{p}} &= -\, p \, n_i \quad \text{on} \quad S_{\mathrm{p}} \end{aligned} \tag{A.3}$$

This stress acting on a cube of material (only a cross-section is visible) is illustrated in . The pores in this porous media are represented by ellipsoids in the .

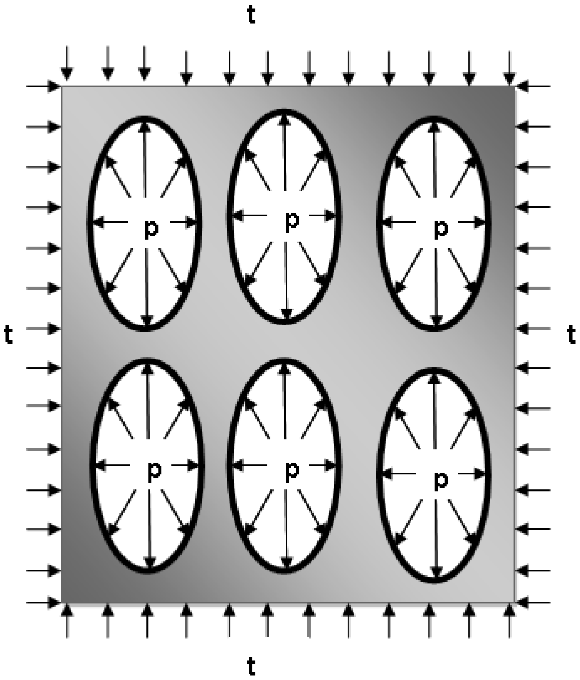

Figure A.3 Picture of total loading for a cube of material representing a mechanically loaded portion of a saturated anisotropic compressible poroelastic medium.

The first key to this proof is to treat the (B.3) as the superposition of two separate stresses

$$\begin{aligned} t_i^0 &= -p\,n_i \quad \text{on} \quad S_0 \\ t_i^{\mathrm{p}} &= -p\,n_i \quad \text{on} \quad S_{\mathrm{p}} \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} t_i^0 &= \sigma_{ij}\,n_j + p\,n_i \quad \text{on} \quad S_0 \\ t_i^{\mathrm{p}} &= 0 \qquad\qquad\quad\; \text{on} \quad S_{\mathrm{p}} \end{aligned} \tag{A.5}$$

The situation for (A.4.) is illustrated in Figure A.4a). It corresponds to the unjacketed conditions where $p = p^{ext}$. It should noted, that the strain in the porous material is equivalent to strain in the matrix material. This means, that uniform straining of the matrix material results in the same straining of the pore space. We can illustrate this clearly by pointing out that the stress relation (A.4) of the solid is achieved by filling the pores with the matrix material. This is pictured in Figure A.4b). Replacing pores with the matrix material has created uniform cube in which the pressure everywhere is the same. Therefore there is no difference in the pressure and strain in Figure A.4a) and those in Figure A.4b. The conclusion that has just been drawn is independent of the shape, size, and connectivity between the pores. Thus, the pores of Figure A.4a) could all be of arbitrary shape and size and they could all be connected as shown in Figure A.4c), but the same pressure p acts everywhere as well as the same homogeneous strain, just as in all three figures.

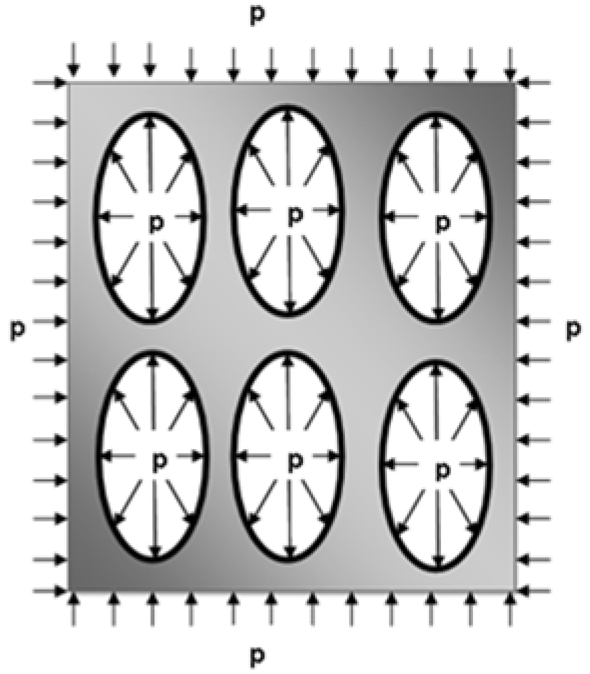

Figure A4a) Picture of total loading for a cube of material with separated pores under unjacketed conditions.

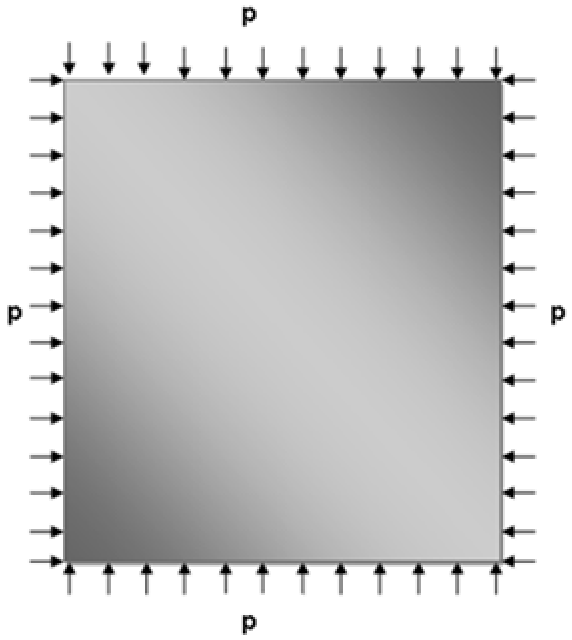

Figure A4b) Picture of total loading for a cube of material without pores.

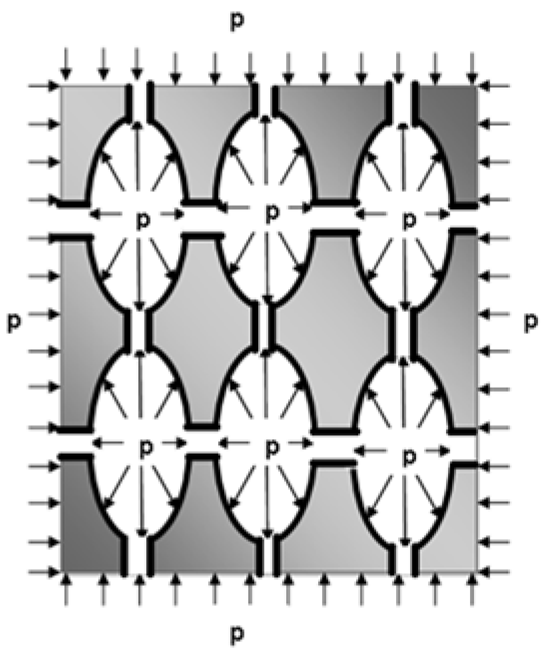

Figure A4c) Picture of total loading for a cube of material with interconnected pores.

The resulting strain for (A.4.) is related to the compliance tensor of the solid s^{s}_{ijkl}

$$\varepsilon^{(1)}_{ij} = -p\, s^{s}_{ijkk} \tag{A.6}$$

The second equation (A.5) looks like relation for stress vector under drained conditions at confining pressure $p = p^{ext}$, but there is also additional stress acting on material. The resulting strain of (A.5) has a form:

$$\varepsilon_{ij}^{(2)} = s_{ijkl}^{m}\left(\sigma_{kl} + p\,\delta_{kl}\right) \tag{A.7}$$

The total strain for dry material is then given by:

$$\varepsilon_{ij}^{m} = \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)} = s_{ijkl}^{m}\,\sigma_{kl} + p\left(s_{ijkk}^{m} - s_{ijkk}^{s}\right) \tag{A.8}$$

The effective stress law is obtained by substituting equation (A.8) into equation (A.1), thus one can obtain:

$$\sigma_{ij}^{\text{eff}} = c_{ijkl}^{m}\left[s_{klmn}^{m}\,\sigma_{mn} + p\left(s_{klmm}^{m} - s_{klmm}^{s}\right)\right] \tag{A.9}$$

or

$$\sigma_{ij}^{\text{eff}} = \sigma_{ij} + p\left(\delta_{ij} - c_{ijkl}^{m}\,s_{klmm}^{s}\right) \equiv \sigma_{ij} + p\,\alpha_{ij} \tag{A.10}$$

where we used relation $\sigma_{ij} = \sigma_{ij}^{\text{eff}} - \alpha\,p\,\delta_{ij}$ from chapter four.

This equation provides the effective stress coefficient α_{ij} in the anisotropic case:

$$\alpha_{ij} = \delta_{ij} - c_{ijkl}^{m}\,s_{klmm}^{s} \tag{A.11}$$

If the solid material is isotropic,

$$s_{klmm}^{s} = \frac{\delta_{kl}}{3\,K_s} \tag{A.12}$$

then relation (A.11) can be rewritten as:

$$\alpha_{ij} = \delta_{ij} - c_{ijkk}^{m}\left(3K_S\right)^{-1} \tag{A.13}$$

Equation (A.13) can be expressed as:

$$\alpha_{I} = \delta_{I} - c_{IJ}^{m}\left(3\,K_s\right)^{-1} \tag{A.14}$$

where α_I has a form:

$$\alpha_{I} = \left(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_5,\right)^{T} \tag{A.15}$$

From (A.15) we can obtain first 6 parameters of relation (6.6).

For fluid-solid coupling Biot's modulus M and bulk modulus $\bar{K}$ for anisotropic case we need to rewrite second of equations (3.17) for anisotropic case as:

$$p = M\left(\varsigma - \alpha_{ij}\,\varepsilon_{kk}^{m}\right) \tag{A.16}$$

using $\varsigma = -\phi\left(\varepsilon_{kk}^{f} - \varepsilon_{kk}^{m}\right)$ under unjacketed conditions we will obtain:

$$\varsigma = -\phi\left(\varepsilon_{kk}^{f} - \varepsilon_{kk}^{m}\right) = -\phi\left(\frac{p}{K_S} - \frac{p}{K_F}\right) \tag{A.17}$$

Strain tensor under unjacketed conditions has form:

$$\varepsilon_{ij}^{m} = -\frac{p\,\delta_{ij}}{3\,K_S} \tag{A.18}$$

Substituting equations (A.17) and (A.18) into (A.16) results in:

$$M = \frac{K_S}{\left[\alpha_{ii} - \phi\left(1 - \frac{K_S}{K_F}\right)\right]} \tag{A.19}$$

Using first three equations from relation (6.6) and introducing them into (A.19), we will obtain:

$$M = \frac{K_S}{\left(1 - \frac{\bar{K}}{K_S}\right) - \phi\left(1 - \frac{K_S}{K_F}\right)} \tag{A.20}$$

where

$$\bar{K} = \frac{1}{9}\left[c_{11} + c_{22} + c_{33} + 2\left(c_{12} + c_{13} + c_{23}\right)\right] \tag{A.21}$$

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